The Riemann-Roch Theorem

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1 Introduction

The goal of this paper is to state and partially prove an analytic version of the Riemann-Roch theorem. Then, using the Riemann-Roch theorem, we will derive the dimension formula for the dimension of the moduli-space of simple J-holomorphic curves.

The paper is structured as follows: In Section 2, we define Cauchy-Riemann operators, which are the main object of study in the Riemann-Roch theorem. Section 3 introduces the Maslov index and boundary Maslov index, and we compute the boundary Maslov index in some simple cases. Finally, we conclude the paper with a proof of the Riemann-Roch theorem in Section 4, and briefly discussing applications in Section 5. In the appendix, we recall some basic notions from Sobolev space theory for the sake of completeness.

2 Cauchy-Riemann Operators

2.1 Smooth Cauchy-Riemann Operators

Throughout this section, we let Σ be a compact Riemann surface with boundary, and $E \to \Sigma$ a smooth complex vector bundle over Σ. Let $j : \Sigma \to \Sigma$ and $J : E \to E$ denote the complex structures on Σ and E respectively. Let $\Omega^k(\Sigma)$ denote the space of smooth complex valued k forms on Σ and $\Omega^{p,q} \subset \Omega^K(\Sigma)$ the subspace of type (p, q) complex valued forms. Similarly, let $\Omega^k(\Sigma, E)$ denote the space of smooth E-valued k-forms on Σ, and $\Omega^{p,q}(\Sigma, E) \subset \Omega^k(\Sigma, E)$ be the subspace of type (p, q) E-valued forms. These are all complex vector spaces.

The complex structure j determines a $\mathbf{C}^*$ action on $\Omega^k(\Sigma)$ and $\Omega^k(\Sigma, E)$ via the map $(a+ib)\cdot\alpha \mapsto a\alpha + bj^*\alpha$. The isotypic components of this action recover the familar decompositions

$$\Omega^k(\Sigma) = \bigoplus_{p+q=k} \Omega^{p,q}(\Sigma)$$

and

$$\Omega^k(\Sigma, E) = \bigoplus_{p+q=k} \Omega^{p,q}(\Sigma, E)$$

Multiplying the complex structure j by -1 swaps $\Omega^{p,q}$ and $\Omega^{q,p}$. Here is the main definition for this section:

Definition 2.1. Let $d : \Omega^0(\Sigma) \to \Omega^1(\Sigma)$ denote the exterior derivative, and let $\pi_2 : \Omega^1(\Sigma) = \Omega^{1,0}(\Sigma) \oplus \Omega^{0,1}(\Sigma) \to \Omega^{0,1}(\Sigma)$ denote the projection. Define an operator $\overline{\partial} : \Omega^0(\Sigma) \to \Omega^{0,1}(\Sigma)$ by the composition $\pi_2 \circ d$. A **Cauchy-Riemann Operator** on $E \to \Sigma$ is a $\mathbf{C}$ linear operator

$$D : \Omega^0(\Sigma, E) \to \Omega^{0,1}(\Sigma, E)$$

which satisfies the Leibnitz rule:

$$D(f\xi) = f(D\xi) + (\overline{\partial} f)\xi$$

for all $\xi \in \Omega^0(\Sigma, E)$ and $f \in \Omega^0(\Sigma)$.

One way to generate an abundance of Cauchy-Riemann operators is via Hermitian structures.

Definition 2.2. A **Hermitian structure** on E is a real inner product $\langle -, - \rangle$ on E such that the complex structure J is orthogonal: $\langle v, Jv \rangle = 0$. A **Hermitian connection** on E is a $\mathbf{C}$-linear operator $\nabla : \Omega^0(\Sigma, E) \to \Omega^1(\Sigma, E)$ satisfying

$$\nabla(f\xi) = f\nabla\xi + (df)\xi$$

and $d\langle \xi_1, \xi_2 \rangle = \langle \nabla \xi_1 \xi_2 \rangle + \langle \xi_1 \nabla \xi_2 \rangle$ for all $f \in \Omega^0(\Sigma)$ and $\xi, \xi_1, \xi_2 \in \Omega^0(\Sigma, E)$. Using a Hermitian connection ∇, we can construct a Cauchy-Riemann operator $\overline{\partial}^{\nabla}$ by

$$\overline{\partial}^{\nabla} \xi := \frac{1}{2}\nabla\xi + \frac{1}{2} J \nabla \xi \circ j$$

Conversely, given a Cauchy-Riemann Operator D, there is a unique Hermitian connection ∇ on E such that $D = \overline{\partial}^{\nabla}$.

2.2 Real Linear Cauchy-Riemann Operators

For the purposes of the Riemann-Roch theorem, we need a slight generalization of Definition 2.1. In particular, we weaken our assumptions on smoothness, and only enforce $\mathbf{R}$-linearity, rather than $\mathbf{C}$-linearity. To begin, define spaces $\Omega^0_F(\Sigma, E)$ and $\Omega^{0,1}_F(\Sigma, E)$ by

$$\Omega^0_F(\Sigma, E) := \{\xi \in \Omega^0(\Sigma, E) : \xi(\partial\Sigma) \subset F\}$$
$$\Omega^{0,1}_F(\Sigma, E) := \{\eta \in \Omega^{0,1}(\Sigma, E) : \eta(T\partial\Sigma) \subset F\}$$

Let $W^{k,q}_F(\Sigma, E)$ be the closure of $\Omega^0_F(\Sigma, E)$ in the Sobolev space $W^{k,q}(\Sigma, E)$, and $W^{k,q}_F(\Sigma, E')$ be the closure of $\Omega^{0,1}_F(\Sigma, E)$ in the Sobolev space $W^{k,q}(\Sigma, E')$, where we use the norm given in Definition A.4, and

$$E' := \Lambda^{0,1} T^*\Sigma \otimes E$$

Here is our definition:

Definition 2.3. Fix a positive integer l, and $p > 1$ such that $lp > 2$. A **real linear Cauchy-Riemann Operator of class** $W^{l-1,p}$ on E is an operator of the form $D = D_0 + \alpha$, where $\alpha \in W^{l-1,p}(\Sigma, \Lambda^{0,1}T^*E \otimes \mathrm{End}_{\mathbf{R}}(E))$, and D_0 is a smooth complex linear Cauchy-Riemann operator on E. Real linear Cauchy-Riemann operators satisfy the equation

$$D(f\xi) = f(D\xi) + (\bar{\partial}f)\xi$$

only for real valued functions f.

Similar to the complex case, we can use a connection ∇ to define D,

$$\bar{\partial}^{\nabla}\xi = \frac{1}{2}(\nabla\xi + J\nabla\xi \circ j)$$

but now the connection need not be Hermitian, so ∇ need not preserve the metric or complex structure on E. However, similar to how we may write a real-linear Cauchy-Riemann operator as the sum of a Cauchy-Riemann operator and a perturbative correction, we may write the connection ∇ as the sum of a hermitian connection ∇_0 and a perturbative correction as follows: Let ∇_0 be any smooth Hermitian connection on E. Then write $\nabla = \nabla_0 + A$, where

$$A \in W^{l-1,p}(\Sigma, T^*\Sigma \otimes_{\mathbf{R}} \mathrm{End}_{\mathbf{R}}(E))$$

such that

$$\bar{\partial}^{\nabla}\xi = \bar{\partial}^{\nabla_0}\xi + \frac{1}{2}(A\xi + JA\xi \circ j)$$

2.3 Gauge Equivalence

The following lemma provides a useful relationship between arbitrary complex linear Cauchy-Riemann operators (i.e of any class) and smooth complex linear Cauchy-Riemann operators, in the special case where we consider line bundles.

Lemma 2.4 (Gauge Equivalence)**.** *Let $E \to S$ be a complex line bundle over a closed Riemann surface, and D be a complex Cauchy-Riemann operator of class L^p over S, for some $p > 2$. Then we may decompose $D = D_0 + \alpha^{0,1}$, where D_0 is smooth and complex linear, and $\alpha \in L^p(\Sigma, T^*\Sigma \otimes_{\mathbf{R}} i\mathbf{R})$. Furthermore, by Hodge theory there is a decomposition $\alpha = \alpha_0 + df + * \, dg$, where $f, g \in W^{1,p}(S, i\mathbf{R})$ and $\alpha_0 \in \Omega^1(S, i\mathbf{R})$ is harmonic (and hence smooth). Defining $u := \exp(-f - ig) \in W^{1,p}(S, \mathbf{C}^*)$, we have $u^{-1}\bar{\partial}u = -(df + * \, dg)^{0,1}$, and hence*

$$u^{-1} \circ D \circ u = D_0 + \alpha_0^{0,1}$$

3 Maslov Indices

In this section we introduce some basic facts about the Maslov and boundary Maslov indices, which are necessary to even state the Riemann-Roch theorem.

3.1 The Maslov Index

Fix an integer n. Let $R_n := GL_n(\mathbf{C})/GL_n(\mathbf{R})$ be the manifold of totally real subspaces of $\mathbf{C}^n$. Define a map $\rho : R_n \to S^1$ by $\rho(X) = \frac{\det(X^2)}{\det(X^*X)}$.

Definition 3.1. Let Γ be any compact oriented 1-manifold without boundary. The **Maslov index** of a map $\Lambda : \Gamma \to R_n$ is

$$\mu(\Lambda) := \deg(\rho \circ \Lambda)$$

For the rest of this section, a 2-manifold is a compact, oriented, 2 manifold Σ with or without boundary.

Definition 3.2. Let M be a 2-manifold. A **decomposition** of M is a pair of submanifolds $A, B \subset \Sigma$ such that $M = A \cup B$ and $A \cap B = \partial A \cap \partial B$.

The notion of a decomposition provides a powerful technique for proving statements about 2-manifolds, or all 2-submanifolds of a given 2-submanifold. Rouhgly speaking, here's how it works. Suppose that we can show

- The theorem holds for a disc.
- Suppose Σ_{02} is our given 2 manifold, and Σ_{01}, Σ_{12} form a decomposition of Σ_{02}. If the theorem holds for two of the Σ_{ij}, then it holds for the third.

We will refer to this induction technique as **pair of pants induction**, it will be necessary to prove the Riemann-Roch theorem.

3.2 The Boundary Maslov Index

We can also have decompositions of vector bundles over an arbitrary Riemann surface.

Definition 3.3. Let Σ be a Riemann surface. A **bundle pair** (E, F) over Σ consists of a vector bundle $E \to \Sigma$ and a totally real subbundle $F \subset E|_{\partial\sigma}$.

Definition 3.4. Let (E, F) be a bundle pair over a Riemann surface Σ. A **decomposition** of (E, F) consists of two bundle pairs: $(E_{01}, F_0 \cup F_1)$ over Σ_{01} and $(E_{12}, F_1 \cup F_2)$ over Σ_{12} such that Σ_{01}, Σ_{12} is a decomposition for Σ.

We can now define the boundary Maslov index, which is of particular interest to us in the case of the Riemann-Roch theorem. Rather than give an explicit definition, the following theorem characterizes the boundary Maslov index uniquely:

Theorem 3.5. *There is a unique operation that assigns to a bundle pair (E, F) an integer $\mu(E, F) \in \mathbf{Z}$ satisfying the following axioms*

(1) *If $\Phi : E_1 \to E_2$ is a bundle isomorphism covering an orientation preserving diffeomorphism $\phi : E_1 \to E_2$, then $\mu(E_1, F_1) = \mu(E_2, \Phi(F_1))$.*

(2) *Direct sums of bundles are additive.*

$$\mu(E_1 \oplus E_2, F_1 \oplus F_2) = \mu(E_1, F_1) + \mu(E_2, F_2)$$

(3) *If (E_{01}, F_{01}) and (E_{12}, F_{12}) is a decomposition of (E, F), then*

$$\mu(E, F) = \mu(E_{01}, F_{01}) + \mu(E_{12}, F_{12})$$

(4) *Let $\Sigma = D$ be the unit disk, and $E = D \times \mathbf{C}$ the trivial bundle. For $z = e^{i\theta} \in S^1$, let $F_z = \mathbf{R}e^{ik\theta/2}$. Then $\mu(D \times \mathbf{C}, F) = k$.*

*The integer $\mu(E, F)$ is called the **boundary Maslov index** of the pair (E, F).*

The following proposition gives a relationship between the Maslov index (Definition 3.1) and the boundary Maslov index. In particular, if Σ has boundary, we have:

Proposition 3.6. *Suppose $\partial\Sigma \neq \emptyset$. If $E = \Sigma \times \mathbf{C}^n$ is a trivial bundle, and $F \subset E|_{\partial E}$ is a totally real subbundle, define $\Lambda(z) := F_z$. Then*

$$\mu(\Sigma \times \mathbf{C}^n, F) = \mu(\Lambda)$$

where the left side is the boundary Maslov index, and the right side is the Maslov index in Definition 3.1.

If Σ has no boundary, we also have a description for the boundary Maslov index. Here it is:

Proposition 3.7. *Let Σ be a Riemann surface without boundary. Then*

$$\mu(E, \emptyset) = 2\langle c_1(E), [\Sigma]\rangle$$

where $c_1(E) \in H^2(\Sigma)$ is the first chern class, and $[\Sigma] \in H_2(\Sigma)$ is the fundamental class.

4 The Riemann-Roch Theorem

In this section, we state and partially prove the Riemann-Roch theorem. First, we fix some notation. Keeping the definitions of Section 2, let $F \subset \partial E$ be a totally real subbundle, and let $E' := \Lambda^{0,1}T^*\Sigma \otimes E$. Let $\langle -, - \rangle$ be a Hermitian form such that $JF = F^\perp$, and dvol a volume form on Σ.

Let D be a real linear Cauchy-Riemann operator. Let D_F be the restriction of D to the subspace $W^{l,p}_F(\Sigma, E)$, so D_F is an operator $D_F : W^{l,p}_F(\Sigma, E) \to W^{l-1,p}(\Sigma, E')$. Given a Hermitian form satisfying $JF = F^\perp$ and a volume form dvol, we define the formal adjoint D^*_F to be the restriction of the formal adjoint D^* of D to the space $W^{l,p}_F(\Sigma, E')$, so D^*_F is an operator $D^*_F : W^{l,p}_F(\Sigma, E') \to W^{l-1,p}(\Sigma, E)$. Here $W^{l,p}_F(\Sigma, E)$ and $W^{l,p}_F(\Sigma, E')$ are defined by

$$W^{l,p}_F(\Sigma, E) := \{\xi \in W^{l,p}(\Sigma, E) : \xi(\partial\sigma) \subset F\}$$

and

$$W^{l,p}_F(\Sigma, E') := \{\eta \in W^{l,p}(\Sigma, E') : \eta(T\partial\Sigma) \subset F\}$$

We can now finally state the Riemann-Roch theorem:

Theorem 4.1 (Riemann-Roch). *Let $E \to \Sigma$ be a complex vector bundle of rank n, and $F \subset E|_{\partial\Sigma}$ a totally real subbundle. Fix a positive integer l and $p > 1$ such that $lp > 2$, and let D be a real linear Cauchy-Riemann operator on E of class $W^{l-1,p}$. Then for every integer $1 \leq k \leq l$, and every real number $q > 1$ such that $\frac{k-2}{q} \leq \frac{l-2}{p}$, the following hold:*

(1) *The operators D_F and D^*_F are Fredholm. Furthermore, the kernels of D_F and D^*_F are independent of k and q, and we have the following duality between the images and kernels of D_F and D^*_F:*

- *We have $\eta \in \operatorname{im} D_F$ if and only if*
$$\int_\Sigma \langle \eta, \eta_0 \rangle \,\mathrm{dvol} = 0$$
for every $\eta_0 \in \ker D^*_F$.
- *We have $\xi \in \operatorname{im} D^*_F$ if and only if*
$$\int_\Sigma \langle \xi, \xi_0 \rangle \,\mathrm{dvol} = 0$$
for evert $\xi_0 \in \ker D_F$.

(2) *The Fredholm index of D_F is $\operatorname{ind}(D_F) = n\chi(\Sigma) + \mu(E, F)$.*

(3) *If E is a complex line bundle ($n = 1$), then D_F is injective only if $\mu(E, F) < 0$, and D_F is surjective only if $\mu(E, F) + 2\chi(\Sigma) > 0$.*

Proof. We only prove assertions (2) and (3). The proof of (1) essentially falls out of the fact that Cauchy-Riemann operators are Fredholm, and we refer to [6, Theorem C.2.3] for the full proof. Thus, we only need to prove (2) and (3). To show (2), we may without loss of generality consider smooth complex linear Cauchy-Riemann operators - every real linear Cauchy-Riemann operator D of class $W^{l-1,p}$ differs from a complex linear smooth Cauchy-Riemann operator by a *compact* operator, and basic Fredholm theory assures that these operators have the same Fredholm index.

Furthermore, by (1) it suffices to consider the case $k = 1$ and $q = 2$. We first prove Theorem 4.1 in the following simpler case:

Theorem 4.2. *Theorem 4.1 holds when Σ is the closed unit disk $\mathbf{D}$ in $\mathbf{C}$ and D is complex linear.*

Before beginning the proof, we state (without proof) the following useful corollary of the first part of the theorem.

Corollary 4.3 (Serre Duality). *Let $E \to \Sigma$ be a complex vector bundle over a compact Riemann surface with boundary, and $F \subset E|_{\partial\Sigma}$ be a totally real subbundle. Let D be a real linear Cauchy-Riemann operator on E of class $W^{l-1,p}$ where l is a positive integer and $p > 1$ such that $lp > 2$. Let $\zeta \in L^r(\Sigma, T^*\Sigma \otimes_{\mathbf{C}} E^*)$, where $r > 1$. Then the following assertions are equivalent.*

- $\int_\Sigma \zeta \wedge D\xi \in \mathbf{R}$ *for every* $\xi \in \Omega^0_F(\Sigma, E)$.
- ζ *is of class* $W^{l,p}$, $D^*\zeta = 0$, *and* $\zeta|_{\partial E}$ *is a section of the subbundle* $T^*\partial E \otimes_{\mathbf{R}} F^*$.

Proof. We refer to [6, Corollary C.1.11] for a proof □

Armed with Serre duality, we now complete the proof of the Riemann-Roch theorem.

Proof. Since the boundary Maslov index is additive over direct sums, and the Fredholm index satisfies the same property, we assume that our vector bundle $E \to \Sigma$ is a complex line bundle. By [6, Corollary C.3.9], we may further assume E is the trivial bundle $E = \mathbf{D} \times \mathbf{C}$, and the totally real subbundle F is defined by

$$F_{e^{i\theta}} = \mathbf{R}e^{ik\theta/2}$$

for $\theta \in \mathbf{R}$ and some integer k. Define spaces

$$X_F := W^{1,2}_F(\mathbf{D}, \mathbf{C})$$

$$Y := L^2(\mathbf{D}, \Lambda^{0,1}T^*\mathbf{D} \otimes \mathbf{C})$$

and let $D_F : X_F \to Y$ be the operator defined by

$$D_F(\xi) = \frac{1}{2}\left(\frac{\partial}{\partial s}\xi + i\frac{\partial}{\partial t}\xi\right)(ds - idt)$$

We now need the following three auxillary lemmas:

Lemma 4.4. *The orthogonal complement of the image of D_F is the space of all $(0,1)$-forms $\zeta d\bar{z}$ where*

- $\zeta : \mathbf{D} \to \mathbf{C}$ *is smooth*
- $\partial_s\zeta - i\partial_t\zeta = 0$
- $\zeta(e^{i\theta}) \in ie^{i\theta + ik\theta/2}\mathbf{R}$.

Proof. Let $\xi \in X_F$ and $\zeta : \mathbf{D} \to \mathbf{C}$ be such that $\partial_s\zeta - i\partial_t\zeta = 0$. We have

$$\begin{aligned}
\int_{\mathbf{D}} \langle \zeta d\bar{z}, D_F(\xi)\rangle dsdt &= \Re \int_{\mathbf{D}} \bar{\zeta}(\partial_s\xi + i\partial_t\xi)dsdt + \Re\int_{\mathbf{D}} \overline{\partial_s\zeta - i\partial_t\zeta}\xi dsdt \\
&= \Re\int_{\mathbf{D}} (\partial_s(\bar{\zeta}\xi) + i\partial_t(bar\zeta\xi))dsdt \\
&= \Re\int_0^{2\pi} e^{i\theta}\overline{\zeta(e^{i\theta})\xi(e^{i\theta})}d\theta
\end{aligned}$$

The right side vanishes for all $\xi \in X_F$ if and only if $\zeta(e^{i\theta}) \in ie^{i\theta+ik\theta/2}\mathbf{R}$. □

The next two lemmas give formulas for the dimensions of the kernel and cokernel of D_F as k varies. We refer to [6] for proofs.

Lemma 4.5. *If $k \geq 1$, then D_F is injective. If $k \geq 0$, then* $\dim\ker D_F = 1 + k$.

Lemma 4.6. *If $k \geq 1$, then D_F is surjective. If $k \leq -2$, then* $\dim\operatorname{coker} D_F = -k - 1$.

If R is an arbitrary complex linear Cauchy-Riemann operator, it is of the form $R = D_F + \alpha$, where $\alpha \in W^{l-1,p}(\mathbf{D}, \Lambda^{0,1}T^*\mathbf{D})$. Taking $k = 0$ in Lemma 4.6, surjectivity implies that there is some $f \in W^{l-1,p}(\mathbf{D}, \mathbf{C})$ such that $f(e^{i\theta}) \in \mathbf{R}$ and $\overline{\partial} f = \alpha$. Define $w := e^f : \mathbf{D} \to \mathbf{C}^*$. Then $w(F) = F$, and $w^{-1}\overline{\partial} w = \alpha$. From this we deduce $w^{-1} \circ D_F \circ w = D_F + \alpha$, and R is injective (resp. surjective) precisely when D_F is injective (resp. surjective). This concludes the proof of Theorem 4.2.

□

We now use pair of pants induction to complete the proof of the Riemann-Roch theorem. Here is our setup: Let $\Sigma_{02} = \Sigma_{01} \cup \Sigma_{02}$ be a decomposition (so in particular we have $\Sigma_{01} \cap \Sigma_{12} = \partial\Sigma_{01} \cap \partial\Sigma_{12}$). Let $\Gamma_1 = \Sigma_{01} \cap \Sigma_{12}$, and define Γ_j so that $\partial\Sigma_{ij} = -\Gamma_i \cup \Gamma_j$, with $\Gamma_i \cap \Gamma_j = \emptyset$.

Let (E_{02}, F_{02}) be a bundle pair over Σ_{02}, with bundle pair decomposition into (E_{01}, F_{01}) and (E_{12}, F_{12}), where $F_i \subset E_{02}|_{\Gamma_i}$ is a totally real subbundle, and $F_{ij} = F_i \cup F_j$. Define spaces X_{ij} and Y_{ij} in a similar manner as above:

$$X_{ij} = W^{1,2}_{F_{ij}}(\Sigma_{ij}, E_{ij})$$
$$Y_{ij} = L^2(\Sigma_{ij}, \Lambda^{0,1}T^*\Sigma_{ij} \otimes_{\mathbf{C}} E_{ij})$$

Let D be a smooth Cauchy-Riemann operator on E_{02}, and denote $D_{ij} : X_{ij} \to Y_{ij}$ for the restriction of D to the spaces above. We have the following nice relationship between the Fredholm indices of the D_{ij}.

Theorem 4.7. *Fix the above notation. Then we have*

$$\operatorname{ind}(D_{02}) = \operatorname{ind}(D_{01}) + \operatorname{ind}(D_{02})$$

Proof. We begin the proof be "normalizing" our given Cauchy-Riemann operator near Γ_1. To begin, let $U \subset \Sigma_{02}$ be a closed tubular neighbourhood of Γ_1, and let $\phi : [-1, 1] \times \mathbf{R}/\mathbf{Z} \to U$ be a diffeomorphism such that $\phi(0, -) = \Gamma_1$. By the first part of Theorem 4.1, we may assume ϕ is holomorphic, and so $i := \phi^* j$ is the standard complex structure on $[-1, 1] \times \mathbf{R}/\mathbf{Z}$ with coordinates $s + it$, where $s \in [-1, 1]$ and $t \in \mathbf{R}/\mathbf{Z}$.
Now, choose a complex trivialization $U \times \mathbf{C}^n \to E|_U$, sending $(z, \zeta) \mapsto \Phi(z)\zeta$. We may without loss of generality assume that $D \circ \Phi = \Phi \circ \overline{\partial}$. To see this, define $A \in \Omega^{0,1}(U, \operatorname{End}_{\mathbf{R}}(\mathbf{C}^n))$ by $A(\zeta_0) := \Phi^{-1} D(\Phi\zeta_0)$, where ζ_0 is the constant map. Let $\zeta : U \to \mathbf{C}^n$ be any smooth map. Then we have $D(\Phi\zeta) = \Phi(\overline{\partial}\zeta + A\zeta)$. Thus we may extend $\Phi A \Phi^{-1} \in \Omega^{0,1}(U, \operatorname{End}_{\mathbf{R}}(E))$ to a global form $B \in \Omega^{0,1}(\Sigma, \operatorname{End}_{\mathbf{R}}(R))$. Replacing D with the Cauchy-Riemann operator $D - B$, we see $D \circ \Phi = \Phi \circ \overline{\partial}$, as desired.

Now we complete the proof of Theorem 4.7. First, define spaces

$$X := W^{1,2}(\Sigma_{01}, E_{01}) \oplus W^{1,2}(\Sigma_{12}, E_{12})$$
$$Y := L^2(\Sigma_{01}, E'_{01}) \oplus L^2(\Sigma_{12}, E'_{12})$$

where $E'_{ij} := \Lambda^{0,1}T^*\Sigma_{ij} \otimes E_{ij}$. Finally, define two subspaces $X_0, X_1 \subset X$ by

$$X_0 := \{(\xi_{01}, \eta_{12}) \in X : \xi_{01}(\Gamma_0) \subset F_0, \xi_{01}(\Gamma_1) \subset F_1, \eta_{12}(\Gamma_1) \subset F_1, \eta_{12}(\Gamma_2) \subset F_2\}$$
$$X_1 := \{(\xi_{01}, \eta_{12}) \in X : \xi_{01}(\Gamma_0) \subset F_0, \eta_{12}(\Gamma_2) \subset F_2, \xi_{01}|_{\Gamma_1} = \eta_{12}|_{\Gamma_1}\}$$

The operator D determines two operators $D_0 : X_0 \to Y$ and $D_1 : X_1 \to Y$, which are both Fredholm, and satisfy

$$\operatorname{ind}(D_0) = \operatorname{ind}(D_{01}) + \operatorname{ind}(D_{12}), \quad \operatorname{ind}(D_1) = \operatorname{ind}(D_{02})$$

To see this, note that we have $D_0 = D_{01} \oplus D_{12}$, and is thus Fredholm - both operators are individually Fredholm by (1) of Theorem 4.1, direct sums of Fredholm operators are Fredholm, and the Fredholm index is additive with respect to direct sums. This shows $\operatorname{ind}(D_0) = \operatorname{ind}(D_{01}) + \operatorname{ind}(D_{12})$. Next, note that the assignment sending $\xi_{02} \in W^{1,2}_{F_{02}}(\Sigma_{02}, E_{02})$ to the pair $(\xi_{02}|_{\Sigma_{01}}, \xi_{02}|_{\Sigma_{12}})$ is an isomorphism, and so the operators D_1 and D_{02} are isomorphic, and hence have the same index.
Now, for $t \in \mathbf{R}/\mathbf{Z}$, define a totally real subspace $\Lambda(t) \subset \mathbf{C}^n$ by

$$\Lambda(t) = \Phi(z_t)^{-1} F_{1,z_t}$$

, where $z_t = \phi(0,t)$. Define

$$I := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

We will construct a smooth map $\Psi : [0,1] \times \mathbf{R}/\mathbf{Z} \to \mathrm{End}_{\mathbf{R}}(\mathbf{C}^n \oplus \mathbf{C}^n)$ satisfying

- $\Psi(s,t)I = I\Psi(s,t)$ for all s,t
- $\Psi(0,t)^{-1}(\Delta) = \Lambda(t) \oplus \Lambda(t)$
- $\Psi(s,t) = \mathrm{id}$ for $1/2 \le s \le 1$.

as follows. First, note that the loop $\Lambda_0 := \Lambda(t) \oplus \Lambda(t)$ of totally real subspaces in $\mathbf{C}^n \oplus \mathbf{C}^n$. It has Maslov index zero, and is thus homotopic to the constant loop

$$\Lambda_1(t) := \Delta := \{(\zeta,\zeta) : \zeta \in \mathbf{C}^n\}$$

Now, choose a smooth homotopy $[0,1] \times \mathbf{R}/\mathbf{Z} \to \mathcal{R}(\mathbf{C}^n \oplus \mathbf{C}^n)$, that sends a pair $(s,t)\Lambda(s,t)$ satisfying $\Lambda(0,t) = \Lambda_0(t)$ and $\Lambda(s,t) = \Lambda_1(t)$ for $1/2 \le s \le 1$. Next, choose a global frame $e_1(s,t), \dots, e_{2n}(s,t)$ of $\Lambda(s,t)$ such that $e_i(s,t) = e_i(1,t)$ for all i and $1/2 \le s \le 1$. Define $\Psi(s,t) \in \mathrm{End}_{\mathbf{R}}(CC^n \oplus \mathbf{C}^n)$ by $\Psi(s,t)e_i(s,t) = e_i(1,t)$ and $\Psi(s,t)Ie_i(s,t) = Ie_(1,t)$ for all i,s,t. Then Ψ satisfies the above three conditions.

To complete the proof of Theorem 4.7, we construct Hilbert space isomorphisms $\Psi_X : X_0 \to X_1$ and $\Psi_Y : Y \to Y$ such that $D_1 \circ \Psi_X - \Psi_Y \circ D_0 : X_0 \to Y$ is compact.
Let Ψ be as above, and write

$$\Psi(s,t) = \begin{pmatrix} A(s,t) & B(s,t) \\ C(s,t) & D(s,t) \end{pmatrix}$$

By the way we constructed Ψ above, A and D are both complex linear and are the identity near $s=1$, while B and C are complex anti-linear amd vanish near $s=1$. Let

$$\begin{aligned} \tilde{\zeta_{01}} &:= A(-s,t)\zeta_{01}(s,t) + B(-s,t)\zeta_{12}(-s,t), & -1 \le s \le 0 \\ \tilde{\zeta_{12}} &:= C(s,t)\zeta_{01}(-s,t) + D(s,t)\zeta_{12}(s,t), & 0 \le s \le 1 \end{aligned}$$

Define $\Psi_X : X_0 \to X_1$ by

$$\Psi_X(\xi_{01},\xi_{02}) := (\tilde{\xi_{01}},\tilde{\xi_{12}})$$

, where $\tilde{\xi_{ij}} = \xi_{ij}$ on $\Sigma_{ij} \setminus U$ and

$$\begin{aligned} \xi_{ij}(\phi(s,t)) &:= \Phi(\phi(s,t))\zeta_{ij}(s,t) \\ \tilde{\xi_{ij}}(\phi(s,t)) &:= \Phi(\phi(s,t))\tilde{} \end{aligned}$$

otherwise. If $(\xi_{01},\xi_{12}) \in X_0$, then for every $z \in \Gamma_1$, we have $\xi_{01}(z), \xi_{12}(z) \in F_{1,z}$. Thus

$$(\zeta_{01}(0,t), \zeta_{12}(0,t)) \in \Lambda(t) \oplus \Lambda(t)$$

for every t. This implies $(\tilde{\zeta_{01}}(0,t), \tilde{\zeta_{12}}(0,t)) \in \Delta$ for every t, and thus $\tilde{\xi_{01}}(z) = \tilde{\xi_{12}}(z)$ for all $z \in \Gamma_1$, which shows that the image of X_0 under Ψ_X is X_1.
We define Ψ_Y in a similar manner. First, define operators

$$\begin{aligned} \tilde{\beta_{01}} &:= A(-s,t)\beta_{01}(s,t) - B(-s,t)\beta_{12}(-s,t), & -1 \le s \le 0 \\ \tilde{\beta_{12}} &:= -C(s,t)\beta_{01}(-s,t) + D(s,t)\beta_{12}(s,t), & 0 \le s \le 1 \end{aligned}$$

Define $\Psi_Y : Y \to Y$ be $\Psi_Y(\eta_{01},\eta_{02}) := (\tilde{\eta_{01}},\tilde{\eta_{12}})$, where similar to above, we define $\tilde{\eta_{ij}} = \eta_{ij}$ on $\Sigma_{ij} \setminus U$, and are otherwise defined by

$$\begin{aligned} \phi^*\tilde{\eta_{ij}} &= (\Phi \circ \phi)\tilde{\beta}_{ij} \\ \phi^*\eta_{ij} &= (\Phi \circ \phi)\beta_{ij}(s,t) \end{aligned}$$

Let $(\xi_{01}, \xi_{12}) \in X_0$ and define $\zeta_{ij}(s,t)$ and $\tilde{\zeta_{ij}}(s,t)$ as above. Using the anti-linearity of B, we compute

$$\partial_s\tilde{\xi}_{01}(s,t) + i\partial_t\tilde{\xi}_{01}(s,t) = A(-s,t)\big(\partial_s\zeta_{01} + i\partial_t\zeta_{01}\big)(s,t) - B(-s,t)\big(\partial_s\zeta_{12} + i\partial_t\zeta_{12}\big)(-s,t)$$
$$+ \big(-\partial_s A + i\partial_t A\big)(-s,t)\zeta_{01}(s,t) + \big(-\partial_s B + i\partial_t B\big)(-s,t)\zeta_{12}(s,t)$$

for $-1 \le s \le 0$ and

$$\partial_s\tilde{\xi}_{12}(s,t) + i\partial_t\tilde{\xi}_{12}(s,t) = -C(s,t)\big(\partial_s\zeta_{01} + i\partial_t\zeta_{01}\big)(-s,t) + D(s,t)\big(\partial_s\zeta_{12} + i\partial_t\zeta_{12}\big)(s,t)$$
$$-\big(\partial_s C + i\partial_t C\big)(s,t)\zeta_{01}(-s,t) + \big(\partial_s D + i\partial_t D\big)(s,t)\zeta_{12}(s,t)$$

for $0 \le s \le 1$. As shown earlier in the proof, we have $D_1(\Psi_X\xi) = \Phi(\phi)\overline{\partial}\tilde{\zeta}$. Thus, $D_1\Psi_X - \Psi_Y D_0$ is a compact operator.

If $D_1\Psi_X - \Psi_Y D_0$ is a compact operator, then D_0 and D_1 must have the same Fredholm index. This concludes the proof of Theorem 4.7, since we have already established $\operatorname{ind}(D_0) = \operatorname{ind}(D_{01}) + \operatorname{ind}(D_{12})$ and $\operatorname{ind}(D_1) = \operatorname{ind}(D_{02})$. □

We can no finish the proof of Theorem 4.1. From Theorem 4.7 and the third axiom for the boundary Maslov index, if follows that if the index formula holds for two of the three surfaces Σ_{ij} in the decomposition, then it holds for the third. Thus the index formula holds by pair of pants induction, and Theorem 4.2.
It only remains to prove the third assertion. We first reduce to the case of a closed Riemann surface. To start, let Σ be a compact connected Riemann surface with nonempty boundary $\Gamma = \partial\Sigma$. Let $\Sigma \times \mathbf{C}$ be the trivial line bundle, and $F \subset \Gamma \times \mathbf{C}$ be a totally real subbundle. We have a map $\lambda : \Gamma \to S^1/\{\pm 1\}$ suhc that $F_z = \lambda(z)\mathbf{R}$ for all $z \in \Gamma$. Note that any section $\xi : \Sigma \to \mathbf{C}$ satisfies $\xi(z) \in F_z$ if and only if $\overline{\xi(z)} = \lambda(z)^{-2}\xi(z)$ for all $z \in \Gamma$.

Let S be the closed Riemann surface $S := \Sigma \times 0, 1/\sim$, where $\Sigma \times 1$ has the reversed complex structure, and $(z,0) \sim (z,1)$ for $z \in \Gamma$. Let $E_0 := (\Sigma \times 0) \times \mathbf{C}$ and $E_1 := (\Sigma \times 1) \times \mathbf{C}$. Define a map $\gamma := \lambda^{-2} : \Gamma \to S^1$, and consider the pullback line bundle $E := E_0 \times_\gamma E_1 \to S$, with the identifications $(z, 0\zeta) \sim (z, 1, \gamma(z)\zeta)$ for all $\zeta \in \Gamma$ and $\zeta \in \mathbf{C}$. A section of E is given by a pair of maps $\zeta_0, \zeta_1 : \Sigma \to \mathbf{C}$ such that $\zeta_1(z) = \gamma(z)\zeta_0(z)$ for all $z \in \Gamma$. The Chern number of E is given by

$$2\langle c_1(E), [S]\rangle = \mu(E_0, F) + \mu(E_1, F) = 2\mu(\Sigma \times \mathbf{C}, F) < 0$$

Now, consider a Cauchy-Riemann operator of the form $\overline{\partial} + \alpha$ on Σ, where

$$\alpha \in L^p(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes \operatorname{End}_{\mathbf{R}}\mathbf{C})$$

for some $p > 2$. This induces a Cauchy-Riemann operatoer D of class L^p on E given by $\overline{\partial} + \alpha$ on $\Sigma \times \{0\}$ and $\partial + \overline{alpha}$ on $\Sigma \times \{1\}$, where $\bar{\alpha}(z,\hat{z}) = \tau\circ\alpha(z,\hat{z})\circ\tau$ with $\hat{z} \in T_z\Sigma$ and $\tau : \mathbf{C} \to \mathbf{C}$ denoting complex conjugation. If $\xi \in W^{1,p}(\Sigma, \mathbf{C})$ satisfies $\overline{\partial}\xi + \alpha\xi = 0$ and $\xi(\partial\Sigma) \subset F$, it gives rise to a section $\zeta \in W^{1,p}(S, E)$ in the kernel of D - $\zeta_0(z) := \xi(z)$ and $\zeta_1(z) = \overline{\xi(z)}$. So it suffices to prove part (3) of Theorem 4.1 for Cauchy-Riemann operators of class L^p on closed Riemann surfaces.

Now, let $E \to S$ be a complex line bundle over a closed Riemann surface S. Let D be a Cauchy-Riemann operator of class L^p over S for some $p > 2$. We will show D is injective when $\mu(E) < 0$. We first consider the case when D is complex linear. In this case, D is guage equivalent to a smooth complex linear Cauchy-Riemann operator D_0 (Lemma 2.4). There is a holomorphic structure on E so that D_0 is our familiar $\overline{\partial}$ operator on E.

Then any element $\xi : S \to E$ in the kernel of D_0 is locally given by the zeroes of a holomorphic function on an open set. If $\xi \neq 0$, then the zeroes of ξ are isolated and have positive index. The chern number of E is the sum of indices of zeroes of a section with isolated zeroes, D_0 has trivial kernel precisely when $\mu(E) - 2c_1(E) < 0$. Gauge equivalent operators have isomorphic kernels, so the kernels of D and D_0 are isomorphic. This concludes the proof in the complex linear case.

To proceed with the proof of the real linear case, we use the following trick. Choose a smooth complex linear Cauchy-Riemann D_0 on the complex line bundle $E \to S$, and write $D = D_0 + a$ for $a \in L^p(S, E'')$, where E'' is defined as

$$E'' = \Lambda^{0,1}T^*S \otimes \operatorname{End}_{\mathbf{R}}(E)$$

Choose some $\xi \in W^{1,p}_F(S,E)$ such that $D\xi = D_0\xi + a\xi = 0$ Define $b \in L^p(S, \Lambda^{0,1}T^*S)$ by

$$b(z,\hat{z}) := \begin{cases} a(z,\hat{z}) & \xi(z) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $D_0 + b$ is a complex linear Cauchy-Riemann operator satisfying $D_0\xi + b\xi = 0$, since by construction $a\xi = b\xi$. What we have shown is that every element in the kernel of a real linear Cauchy-Riemann operator is also in the kernel of a complex linear Cauchy-Riemann operator of class L^p on the same bundle. This shows that every real linear Cauchy-Riemann operator on a complex line bundle $E \to S$ with negative Chern number is injective. To prove the surjectivity asusmption, by Serre duality (Corollary 4.3), the cokernel of D is isomorphic to the kernel of a Cauchy-Riemann operator on the pair $(\Lambda^{0,1}T^*\Sigma \otimes_{\mathbf{C}} E^*, T\partial\Sigma \otimes_{\mathbf{R}} (E/F)^*)$, with boundary Maslov index $-\mu(E,F) - 2\chi(\Sigma)$. Thus, the cokernel vanishes precisely when $\mu(E,F) + 2\chi(\Sigma) > 0$. This concludes the proof. □

5 Applications

In this section we state an important application of the Riemann-Roch theorem.

5.1 Moduli Spaces of J-Holomorphic Curves

Let (M^{2n}, ω) be a symplectic manifold, and (Σ, j) a compact Riemann surface, with an ω-tame almost complex structure J. Consider the equation

$$\overline{\partial}_J(u) = 0$$

where the operator $\overline{\partial}_J$ is defined by

$$\overline{\partial}_J(u) := \frac{1}{2}(du + J \circ du \circ j)$$

Given a homology class $A \in H_2(M; \mathbf{Z})$, we define the **moduli space** of solutions representing the class A by

$$\mathcal{M}(A, \Sigma, J) := \{u \in C^\infty(M) : J \circ du = du \circ j, [u] \in A\}$$

and

$$\mathcal{M}^*(A, \Sigma, J) := \{u \in \mathcal{M}(A, \Sigma, J) : u \text{ is simple}\}$$

We are interested in the dimension of this moduli-space. It turns out that it will be related to the index of of a Cauchy-Riemann operator D_u, which we will define shortly. First, we recall that we may realize $\mathcal{M}(A, \Sigma, J)$ above as the zero set of some section of some infinite dimensional vector bundle.
Let $\mathcal{B} \subset C^\infty(\Sigma, M)$ denote the space of all smooth maps $u : \Sigma \to M$ that represent the homology class A. This is an infinite dimensional manifold, whose tangent space at $u \in \mathcal{B}$ is given by

$$T_u\mathcal{B} = \Omega^0(\Sigma, u^*TM)$$

Consider the infinite dimensional vector bundle $\mathcal{E} \to \mathcal{B}$ whose fiber at u is the space

$$\mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*TM)$$

of smooth antilinear 1-forms with values in u^*TM. The complex antilinear part of du defines a section $\mathcal{S} : \mathcal{B} \to \mathcal{E}$ given by $\mathcal{S}(u) = (u, \overline{\partial}_J(u))$. Then it follows that the moduli space $\mathcal{M}(A, \Sigma, J)$ is the zero set of this section.

Given $u \in \mathcal{M}^*(A, \Sigma, J)$ as above, define an operator $D_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$ for the composition of the differential $d\mathcal{S}(u)T_u\mathcal{B} \to T_{(u,0)}\mathcal{E}$ with the projection

$$T_(u, 0)\mathcal{E} = T_u\mathcal{B} \oplus \mathcal{E}_u \to \mathcal{E}_u$$

Definition 5.1. We call the operator D_u defined above the **vertical differential** of the section $\mathcal{S}$ at u.

In local coordinates s on M and t on M, a J-holomorphic curve $u : \mathbf{C} \to RR^{2n}$ satisfies

$$\partial_s u + J(u)\partial_t u = 0$$

and a vector field along u is a map $\xi : \mathbf{C} \to \mathbf{R}^{2n}$. Thus, locally we may write D_u by differentiating the above equation in the direction of ξ. This gives

$$D_u \xi = \frac{1}{2}\left(\partial_s \xi + J(u)\partial_t \xi + \partial_\xi J(u)\partial_t u\right) ds - \frac{1}{2}J(u)\left(\partial_s \xi + J(u)\partial_t \xi + \partial_\xi J(u)\partial_t u\right) dt$$

Since u is J-holomorphic, we conclude

$$D_u \xi = \overline{\partial}_J \xi - \frac{1}{2}(J\partial_\xi J)(u)\partial_J(u)$$

This shows that D_u is a Cauchy-Riemann operator. In particular, it is Fredholm. By the Riemann-Roch theorem, its index is given by

$$D_u) = n(2-2g) + 2c_1(u^*TM)$$

It follows that the dimension of the moduli space is given by

$$\dim \mathcal{M}^*(A, \Sigma, J) = n(2-2g) + 2c_1(u^*TM)$$

Remark 5.2. We can define the operator D_u for arbitrary smooth maps $u : \Sigma \to M$ with a little more work. Now, D_u will depend on a choice of splitting of the tangent space $T_{(u, \overline{\partial}_J(u))}$, which depends on a connection on TM. We refer to [6, 3.1] for the exact details.

A Sobolev Spaces

The goal of this section is to introduce Sobolev spaces. We will be by no means comprehensive, and refer to [1] for further details. Througout this section, let $\Omega \subset \mathbf{R}^n$ be an open subset. Let $C_0^\infty(\Omega)$ be the space of smooth compactly supported functions on Ω, and $C^\infty(\overline{\Omega})$ be the space of restrictions of smooth functions on $\mathbf{R}^n$ to $\overline{\Omega}$.

A.1 Sobolev Spaces on Euclidean Space

Definition A.1. Let $u : \Omega \to \mathbf{R}$ be a locally integrable function, and $\nu = (\nu_1, \dots, \nu_n)$ a multi-index. A locally integrable function $u_\nu : \Omega \to \mathbf{R}$ is a **weak derivative** of u corresponding to ν if for every test function $\phi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega u(x)\partial^\nu \phi(x)dx = (-1)^{|\nu|}\int_\Omega u_\nu(x)\phi(x)dx$$

A weak derivative, if it exists, is uniquely determined by u almost everywhere, so we may speak of the weak derivative of u corresponding to ν, and write $\partial^\nu u := u_\nu$.

Using weak derivatives, we can define Sobolev spaces.

Definition A.2. Fix some non-negative integer k and some number $1 \le p \le \infty$. The **Sobolev space** $W^{k,p}(\Omega)$ is defined as the space of all functions $u \in L^p(\Omega)$ such that the weak derivative $\partial^\nu u$ exists and is p-integrable for every ν such that $|\nu| \le k$. When $1 \le p < \infty$, we define the $W^{k,p}$ norm of a function $u \in W^{k,p}(\Omega)$ by

$$||u||_{k,p} = \left(\int_\Omega \sum_{|\nu| \le k} |\partial^\nu u(x)|^p dx\right)^{1/p}$$

The space $W^{k,p}_{\text{loc}}(\Omega)$ is the space of locally p-integrable functions $u : \Omega \to \mathbf{R}$ such that for every precompact open $Q \subset \Omega$, we have $u \in W^{k,p}(Q)$. The space $W_0^{k,p}$ is defined to be the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$, it is the completion of $C_0^\infty(\Omega)$ with respect to the $W^{k,p}$ norm.

In the following proposition, we recall (without proof) some basic facts about Sobolev spaces. All proofs can be found in [1], §5.

Proposition A.3. *Let Ω be as above. Then we have*

- $W^{k,p}(\Omega)$ *is a Banach space.*
- $W^{k,p}(\Omega)$ *is reflexive when* $1 < p < \infty$ *and separable when* $1 \leq p < \infty$.
- *When* $k = 2$, *the Sobolev space* $H^k(\Omega) := W^{2,k}(\Omega)$ *is a Hilbert space.*

A.2 Sobolev Spaces on Manifolds

For our purposes, it will be useful to have a notion of Sobolev regularity on arbitrary smooth manifolds. More specifically, we have the following definition.

Definition A.4. Let M^n be a smooth compact manifold and $\pi : E \to M$ a smooth vector bundle. A section $s : M \to E$ is of **class** $W^{k,p}$ if all of the coordinate representations of S are in $W^{k,p}$. To define a norm on the space of $W^{k,p}$ sections, take the sum of the $W^{k,p}$ norms over finitely many coordinate charts that cover M.

More generally, if X^n and M aer smooth closed manifolds, with $kp > n$. Then, we may define the Sobolev space $W^{k,p}(X, M)$ as the space of continous functions $u : X \to M$ that are represented by $W^{k,p}$ functions in local coordinate charts.

This notion is coordinate-independent, see [6, 561] Remarks B.1.23 and B.1.24.

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