Étale Fundamental Groups

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December 20, 2024

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1 Introduction

In algebraic topology, one associates to each space X a group $\pi_1(X)$, called the fundamental group of X, defined by considering homotopy classes of loops $[0,1] \to X$. While it's definition is relatively simple, the fundamental group is incredibly useful at classifying topological spaces.
One wishes for an analogous theory for spaces that occur in algebraic geometry, but unfortunately, naively applying the fundamental group construction to a scheme is not as fruitful as in the case for general topological spaces - for example, any space with a generic point is contractible, and hence have trivial fundamental groups - which is clearly a problem when more algebro-geomtric spaces are considered. Instead, we consider the following alternative description of the fundamental group. Instead of just looking at the space X itself, we consider *coverings* of X.

Definition 1.1. Let X, Y be spaces, and $f : Y \to X$ a continous map. We say $f : X \to Y$ is a **trivial covering** if we have a homeomorphism $Y \cong X \times E$ for some discrete set E and f can be described as the projection $X \times E \to X$. A map $g : Y \to X$ is a **covering** if X can be covered by open sets U_i such that for each i, we have $f^{\varepsilon} U_i \to U_i$ is a trivial covering. A covering is **finite** if for all $x \in X$, the cardinality of the set $f^{-1}(x) \subset Y$ is finite.
A **morphism** from a covering $f : Y \to X$ to a covering $g : Z \to X$ is a continous map $h : Y \to Z$ such that $f = gh$. Thus, we have categories of (finite) coverings of a given space X.

The fundamental group $\pi_1(X)$ then classifies coverings of X in the following sense. There is a one-to-one correspondence between coverings of X, up to isomorphism, and sets that are provided with a $\pi_1(X)$ action, up to isomorphism. Moreover, this identification respects morphisms - that is, we have the following equivalence of categories:

Theorem 1.2. *Let X be a connected, locally path-connected, and semilocally simply connected space. Then there is an equivalence of categories between the category of coverings of X and the category of sets equipped with a $\pi_1(X)$ action.*

When we weaken the conditions on our space X, and consider only *finite* covers of X, we have the following:

Theorem 1.3. *Let X be a connected space. Then there is a profinite group $\hat{\pi}(X)$, uniquely determined up to isomorphism, such that the category of finite coverings of X is equivalent to the category of finite sets equipped with a continous $\hat{\pi}(X)$-action.*

Theorem 1.3 will serve as a model for many of the theorems proved in this paper - we will show many categories are equivalent to the category $\text{FinSet}(\pi)$ of finite sets equipped with a continous action of some profinite group π. The ultimate goal of this paper is to develop a theory of fundamental groups for schemes, motivated by Theorem 1.3. The main result of this paper is as follows:

Theorem 1.4. *Let X be a connected scheme. Then there a profinite group $\pi(X)$, determined uniquely up to isomorphism, such that the category FEt_X of finite étale coverings of X and the category of finite sets equipped with a continous $\pi(X)$-action are equivalent.*

The paper is structured as follows: In section 2, we treat infinite Galois theory from a more categorical perspective to motivate the constructions made throughout the rest of the paper. Sections 3, 4, and 5 form the heart of this paper. In section 3, we introduce Galois categories, which are central to the rest of the paper. In 4, we recall some basic facts about projective modules and algebras, which serve as an affine model for the general theory we develop in section 5. In section 5, we prove Theorem 1.4, and conclude in section 6 by computing examples.

2 Infinite Galois Theory

In this section we recall basic infinite Galois theory, for the purpose of motivating the constructions of Galois categories.

2.1 Profinite groups

The classical Galois theory of finite extensions assigns to each finite extension of fields L/K a finite Galois group $\text{Gal}(L/K)$ from which one deduces many important facts about the extension L/K itself. A natural idea is to extend this theory to the case of infinite extensions. To accomplish this, we need the theory of profinite groups.

Definition 2.1. Let $I, \geq$ be a partially ordered set. We say I is **directed** if for any $i, j \in I$, there is some $k \in I$ such that $k \geq i$ and $k \geq j$.

Definition 2.2. A **projective system** in a category $\mathcal{C}$ consists of the following data:

- A directed partially ordered set I.
- A collection of objects $(S_i)_{i \in I}$ in $\mathcal{C}$
- A collection of morphisms $(f_{ij} : S_i \to S_j)_{i,j \in I, i \geq j}$ such that
  - $f_{ii} = \text{id}_{S_i}$ for all $i \in I$.
  - $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \geq j \geq k \in I$.

Any partially ordered set can be viewed naturally as a category. Taking this perspective, the data of a projective system is precisely the data of a diagram over the indexing poset category. In particular, we can take the limit of this diagram.

Definition 2.3. Given a projective system as above, the **projective limit**, denoted $\varprojlim_I S_i$ is the limit of the data of the projective system S_i, f_i, viewed as a diagram over I. When $\mathcal{C}$ is the category of sets, groups, modules, topological spaces, etc, we have the following explicit description of the projective limit.

$$\varprojlim_I S_i = \{(x_i)_{i \in I} \in \prod_{i \in I} S_i \, : \, f_{ij}(x_i) = x_j \text{ for all } i, j \in I \text{ with } i \geq j\}$$

Definition 2.4. Let $I, (\pi_i)_{i \in I}, (f_{ij})$ be a projective system of *finite* groups and group homomorphisms. Furthermore, endow each π_i with the discrete topology, so the homomorphisms $f_i j$ are continous. Since each π_i is finite, the discrete topology makes each π_i compact. The projective limit $\pi := \varprojlim_I \pi_i$ is a topological group, called a **profinite**. It is a closed subgroup of $\prod_i \pi_i$, and is hence compact. Furthermore, it is totally disconnected - the only connected components are the singleton sets. Conversely, it can be shown that *any* compact, totally disconnected topological group arises as the projective limit of a projective system of finite groups.

Profinite groups show up everywhere - they are the basic object of study in infinite Galois theory, which we will see shortly. Here are some examples.

Example 2.5. Let G be any group, and let I be the collection of all finite index normal subgroups of G, with $N \geq N'$ when $N \subset N'$. The collection $(G/N)_{N \in I}$ is a projective system of finite groups, with the maps $G/N \to G/N'$ being the projection maps. The projective limit

$$\hat{G} := \varprojlim_I G/N$$

is a profinite group, called the **profinite completion** of G.

Example 2.6. Let p be a prime number, and let I be the set of all integers, with the usual order. Then $(\mathbf{Z}/p^n\mathbf{Z})_{n>0}$ with the projection maps is a projective system, and the limit

$$\mathbf{Z}_p := \varprojlim \mathbf{Z}/p^n\mathbf{Z}$$

is a profinite group. It is actually a profinite ring, the ring of p-adic integers.

Definition 2.7. If π is a profinite group, a π-set is a set E equipped with a continous π-action - that is a continous $\pi \times E \to E$, where E is given the discrete topology, that also is a group action. A morphism of π-sets is a continous map $f : E \to E'$ such that $f(\sigma e) = \sigma f(e)$ for all $\sigma \in \pi$. We denote the category of *finite* π-sets and morphisms of π-sets by $\mathrm{FinSet}(\pi)$.

2.2 Separable Algebras

Let A be a ring, and B a A-algebra that is finitely generated and free as an A-module. For any $b \in B$, there is a natural A-linear map $m_B : B \to B$, sending $x \mapsto bx$. We say the trace of an element $b \in B$ is $\mathrm{Tr}(b) = \mathrm{Tr}(m_b)$. This gives an A-linear map $\mathrm{Tr} : B \to A$, and for $a \in A$ we have $\mathrm{Tr}(a) = \mathrm{rank}_A(B) \cdot a$.

Definition 2.8. The A-module $\mathrm{Hom}_A(B, A)$ is free over A, and has the same rank as B. Define an A-linear map $\phi : B \to \mathrm{Hom}_A(B, A)$ by $\phi(x)(y) = \mathrm{Tr}(xy)$ for $x, y \in B$. If the map ϕ is an isomorphism, we say B is **separable** over A, or say B is a free separable A-algebra. We denote the category of free separable A-algebras by SAlg_A.

Example 2.9. For any ring A, the A-algebra $A^{\oplus n}$ is a free separable A-algebra. If $A = \mathbf{Z}$ or A is an algebraically closed field, then there are no others.

The following two results characterize free separable algebras over a field k.

Lemma 2.10. *Let B be a finite dimensional algebra over a field k. Then*

$$B \cong \prod_{i=1}^{t} B_i$$

where t is some positive integer and B_i are local rings with nilpotent maximal ideals.

Proof. First, consider the case when B is a domain. Then for any nonzero $b \in B$, the map $x \mapsto bx$ is injective and surjective, so $b \in B^\times$. So if B is a domain, it is a field. It follows that any prime ideal of B is maximal. Given distinct maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots \mathfrak{m}_n$, the Chinese remainder theorem shows that the natural map

$$B \to \prod_{i=1}^{n} B/\mathfrak{m}_i$$

is surjective. Thus $n \le \dim_K B$, and B has finitely many maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_t$. The intersection of all the maximal ideals is the nilradical of B, since every prime is maximal. B is noetherian, so $\mathrm{nil}(B)$ is nilpotent, so for $N >> 0$, we have $\prod_{i=1}^{t} \mathfrak{m}_i^N = 0$. Applying the Chinese remainder theorem again, we have an isomorphism $B \cong \prod_{i=1}^{t} B/\mathfrak{m}_i^N$. Each $B/\mathfrak{m}_i^N$ is local since $\mathfrak{m}_i/\mathfrak{m}_i^N$ is its only maximal ideal, which is nilpotent. Taking $B_i := B/\mathfrak{m}_i^N$ completes the proof. □

The above lemma can be slightly generalized to consider algebras over an Artinian ring, with a slightly more complicated proof, but for our cases we will only need the statement for k-algebras.

Lemma 2.11. *Let k be a field with algebraic closure $\overline{k}$. Let B be a finite dimensional k-algebra, and $\overline{B} := B \otimes_k \bar{k}$. The following are equivalent:*

- *B is separable over k.*
- *$\overline{B}$ is separable over $\bar{k}$.*
- *$\overline{B} \cong \overline{k}^n$ as $\overline{k}$ algebras for some $n \ge 0$.*
- *$B \cong \prod_{i=1}^{t} B_i$ as k-algebras, where each B_i is a finite separable extension of k.*

Proof. (1) $\iff$ (2): Let $w_1, \dots, w_n$ be a k-basis for B. Then $w_1 \otimes 1, \dots, w_n \otimes 1$ is a $\overline{k}$-basis for $\overline{B}$. T This gives the following commutative diagram:

$$\begin{array}{ccc} B & \hookrightarrow & \overline{B} \\ \downarrow \mathrm{Tr}_{B/k} & & \downarrow \mathrm{Tr}_{\overline{B}/\overline{k}} \\ k & \hookrightarrow & \overline{k} \end{array}$$

This shows $\mathrm{Tr}_{B/k}(w_i w_j) = \mathrm{Tr}_{\overline{B/k}}((w_i \otimes 1)(w_j \otimes 1))$. The equivalence follows after noting that a finite free A-algebra B is separable over A if and only if $\det(Tr(w_i w_j)_{1,\le i,j \le n}) \in A^{\times}$.

(3) $\Longrightarrow$ (2): Follows from Example 2.9.
(2) $\Longrightarrow$ (3): Applying Lemma 2.10 to $\overline{k}$ and $\overline{B}$, we see that $\overline{B} \cong \prod_{j=1}^{u} C_j$ for local $\overline{k}$-algebras with nilpotent maximal ideals $\mathfrak{m}_j$, and each C_j separable over $\overline{k}$. For some fixed j, let $\phi : C_j \to \overline{k}$ be some $\overline{k}$-linear function. Then there is some $c \in C_j$ such that $\phi(x) = \mathrm{Tr}(cx)$. Taking $x \in \mathfrak{m}_j$, we see that $\mathfrak{m}_j \subset \ker \phi$, since nilpotent maps are traceless over a field. This is true for each ϕ, so $\mathfrak{m}_j = \{0\}$ and C_j is a field. Since C_j is finite over $\overline{k}$, we conclude $C_j = \overline{k}$.
(4) $\Longrightarrow$ (3): Write $B_i \cong k(\beta_i) \cong k[x]/(f_i)$ with $f_i \in k[x]$ separable and irreducible. Then $\overline{B} \cong \overline{k}[x]/(f_i)$. Since f_i splits into linear factors $x - \alpha_{ij}$in $\overline{k}[x]$, the Chinese remainder theorem gives

$$\overline{B_i} \cong \prod_j \overline{k}[x]/(x - \alpha_{ij}) \cong \overline{k}^{\deg(f_i)}$$

(3) $\Longrightarrow$ (4): Using Lemma 2.10, write $B \cong \prod_{i=1}^{t} B_i$. For each $b \in B$, we have an isomorphism $k[b] \cong k[x]/(f_b)$ for some nonzero $f_b \in k[x]$. Tensoring up, we get an injective map $\overline{k}[x]/(f_b) \to \overline{B}$. Assuming (3), $\overline{k}[x]/(f_b)$ has no nilpotent elements, so f_b is separable. Thus all the B_i are fields. For $b = (b_1, \dots, b_t) \in B$, the polynomial f_b is the least common multiple of all of the irreducible polynomials of the b_i over k, which are all separable. Thus all the B_i are separable field extensions of k. □

2.3 An Equivalence

Let K be a field, K_s be its separable closure, and $\pi := \mathrm{Gal}(K_s/K)$, which we will refer to as the **absolute Galois group** of K. Let B be a free separable K-algebra, and consider the set $\mathrm{Hom}_{Alg}(B, K_s)$ of K-algebra homomorphisms

$B \to K_s$. There is a natural π-action on this set - given $\sigma \in \pi$ and $g : B \to K_s$, we define the action by $\sigma \cdot g := \sigma \circ g$. Since $\sigma \circ G$ is also a K-algebra homomorphism, we have a well-defined action on $\mathrm{Hom}_{Alg}(B, K_s)$.

The preceding paragraph hints at the existence of a functor from the category of separable K-algebras to the category of sets equipped a π-action. In fact, we will shortly see that we actually have a functor $\mathrm{SAlg}_K \to \mathrm{FinSet}(\pi)$. We need the following lemma.

Lemma 2.12. *With the same notation as above, the set* $\mathrm{Hom}_{Alg}(B, K_s)$ *is finite, and the* π*-action on it is continous.*

Proof. As in Lemma 2.11, write $B = \prod_{i=1}^{t} B_i$.We may thus identify each B_i as a subfield of K_s, and thus as the fixed field $K_s^{\pi_i}$ of some open subgroup $\pi_i \subset \pi$, for $1 \leq i \leq t$. We therefore have a decomposition

$$\mathrm{Hom}_{Alg}(B, K_s) \cong \coprod_i \mathrm{Hom}_{Alg}(K_s^{\pi_i}, K_s)$$

Each $\mathrm{Hom}_{Alg}(K_s^{\pi_i}, K_s)$ is the set of field homomorphisms $K_s^{\pi_i} \to K_s$ that fix K, which can be identified naturally with π/π_i that respects that π-action. Therefore, we have

$$\mathrm{Hom}_{\mathrm{Alg}}(B, K_s) \cong \coprod_{i=1}^{t} \pi/\pi_i$$

Since the π_i are open, each π/π_i is finite, and thus $\mathrm{Hom}_{\mathrm{Alg}}(B, K_s)$ is a finite set on which π acts continously. □

Corollary 2.13. *The assignment* $B \mapsto \mathrm{Hom}_{\mathrm{Alg}}(B, K_s)$ *defines a functor* $F : \mathrm{SAlg}_K \to \mathrm{FinSet}(\pi)$*.*

Proof. The fact that $\mathrm{Hom}_{\mathrm{Alg}}(B, K_s)$ is a finite set equipped with a π-action is the content of Lemma 2.12. It remains to show that there is a corresponding assignment of morphisms. For this, we take the usual action of any Hom functor - given a morphism $B \to C$ of separable K algebras, we have a morphism $\mathrm{Hom}_{\mathrm{Alg}}(C, K_s) \to \mathrm{Hom}_{\mathrm{Alg}}(B, K_s)$. □

We have a similar construction in the opposite direction as well. Let E be a finite π-set, and consider the set $\mathrm{Hom}_\pi(E, K_s)$ of π-equivariant maps $E \to K_s$. We can endow $\mathrm{Hom}_\pi(E, K_s)$ with the structure of a K-algebra by defining addition, multiplication, and scaling pointwise. We fist show that this is indeed a free separable K-algebra.

Lemma 2.14. $\mathrm{Hom}_\pi(E, K_s)$ *is a finite free separable* K*-algebra.*

Proof. First, decompose $E = \coprod_{i=1}^{t} E_i$ into its set of orbits under π. Then we have an isomorphism

$$Hom_\pi(E, K_s) \cong \prod_i \mathrm{Hom}_\pi(E_i, K_s)$$

Each E_i is isomorphism to π/π_i as a π-set for some open subgroup $\pi_i \subset \pi$. Every morphism of π-sets $g : \pi/\pi_i \to K_s$ is given by $g(\sigma) = \sigma(a)$ for some $a \in K_s$. Conversely, given $a \in K_s$, g is a well defined map if and only if $a \in K_s^{\pi_i}$. Therefore we have $\mathrm{Hom}_\pi(\pi/\pi_i) \cong K_s^{\pi_i}$, so we conclude

$$Hom_\pi(E, K_s) \cong \prod_{i=1}^{t} K_s^{\pi_i}$$

which is a finite dimensional vector space. □

Corollary 2.15. *The assignment* $E \mapsto \mathrm{Hom}_\pi(E, K_s)$ *defines a contravariant functor* $\mathrm{FinSet}(\pi) \to \mathrm{SAlg}_K$.

Proof. Again, that $\mathrm{Hom}_\pi(E, K_s)$ is indeed a free separable K algebra is the content of Lemma 2.14. We need to define an action on morphisms, so again we take the induced map by any Hom functor - given a morphism $E \to D$ of π-sets, we get a morphism $\mathrm{Hom}_\pi(D, K_s) \to \mathrm{Hom}_\pi(E, K_s)$. □

We now have enough machinery to state and prove the main theorem of this section:

Theorem 2.16. *Let K be a field, and K_s be its separable closure. Let π denote the Galois group* $\mathrm{Gal}(K_s/K)$*. Finally, let $F : \mathrm{SAlg}_K \to \mathrm{FinSet}(\pi)$ and $G : \mathrm{FinSet}(\pi) \to \mathrm{SAlg}_K$ be the two functors defined above. Then F and G define an equivalence of categories* $\mathrm{FinSet}(\pi) \cong \mathrm{SAlg}_K$.

Proof. Now we verify naturality. For a free separable K-algebra B, define

$$\theta_B : B \to GF(B)$$

by $\theta_B(b)(g) = g(b)$. This is a well defined homomorphism of K-algebras. Given a homomorphism $f : B \to C$ of K-algebras, we claim the following square commutes:

$$\begin{CD} B @>{f}>> C \\ @V{\theta_B}VV @VV{\theta_C}V \\ GF(B) @>{GF(f)}>> GF(C) \end{CD}$$

Indeed, we have $(\theta_C \circ f)(b)(g) = \theta_C(f(b))(g) = g(f(b))$ and $GF(f)(\theta_B(b))(g) = (\theta_B(b) \circ F(f))(g) = \theta_B(g \circ f) = g(f(b))$. Next, note that for $B = \prod_{i=1}^{t} K_s^{\pi_i}$, the map θ_B is an isomorphism, and thus θ_B is an isomorphism for all B. This shows the functor GF is naturally isomorphic to the identity functor on SAlg_K.

Similarly, for a finite π-set E, define $\eta_E : E \to FG(E)$ by $\eta_E(e)(g) = g(e)$. This is a well defined morphism of π-sets. Given a map $f : E \to D$ of π-sets, we claim the following square commutes:

$$\begin{CD} E @>{f}>> D \\ @V{\eta_E}VV @VV{\eta_D}V \\ FG(B) @>{FG(f)}>> FG(C) \end{CD}$$

Indeed, we have $(\eta_D \circ f)(e)(g) = \eta_D(f(e))(g) = g(f(e))$ and $FG(f)(\eta_E(e))(g) = (\eta_E(e) \circ G(f))(g) = \eta_E(e)(g \circ f) = g(f(b))$. For $E = \coprod_{i=1}^{t} \pi/\pi_i$ the map η_E is an isomorphism, so this holds for all E. This completes the proof of Theorem 2.16.

□

3 Galois Categories

In this section we introduce and collect some basic facts about Galois categories. Throughout, let $\mathcal{C}$ be a category. Before providing the definition of a Galois category, we generalize the notion of a group action to the level of categories. The main result of this section proves that any Galois category is equivalent to the category of π-sets for some uniquely determined profinite group π.

Definition 3.1. Let c be an object of C, and $G \subset \mathrm{Aut}(c)$ a finite subgroup of the group of automorphisms of c in $\mathcal{C}$. The **quotient** of c by G is a an object $c/G \in \mathcal{C}$, together with a morphism $p : c \to c/G$ satisfying $p = p\sigma$ for all $\sigma \in G$, that is universal in the following sense: If $f : c \to c'$ is another morphism in $\mathcal{C}$ such that $f = f\sigma$ for all $\sigma \in G$, then there is a unique map $\psi : c/G \to c'$ such that $f = \psi p$.

$$\begin{array}{ccc} c & \xrightarrow{\quad f \quad} & c' \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle \exists!\psi} & \\ & c/G & \end{array}$$

As with other universal constructions, the quotient is unique up to unique isomorphism, if it exists.

With this defintion, we can define Galois categories.

Definition 3.2. Let $\mathcal{C}$ be a category, and $F : \mathcal{C} \to \mathrm{FinSet}$ a functor. The pair $(\mathcal{C}, F)$ is a **Galois category** if it satisfies the following additional properties:

(1) $\mathcal{C}$ has a terminal object, and all pullbacks.

(2) $\mathcal{C}$ has finite coproducts, and for any object $X \in \mathcal{C}$ and finite subgroup $G \subset \mathrm{Aut}(X)$, the quotient X/G exists.

(3) Any morphism $f \in \mathcal{C}$ factors as $f = f'f''$, where f'' is an epimorphism and f' is a monomorphism. Furthermore, any monomorphism $m : X \to Y$, there is an object Z and a morphism $q : Z \to Y$, such that $Y = X \coprod Z$.

(4) F preserves terminal objects and pullbacks.

(5) F commutes with finite coproducts, preserves epimorphisms, and commutes with passage to the quotient.

(6) F reflects isomorphisms.

We often refer to a Galois category as just the category $\mathcal{C}$, and call F the **fundamental functor** of $\mathcal{C}$.

We check that the above definition is meaningful, with the following two examples.

Example 3.3. The category FinSet is a Galois category. We verify the six axioms above in order.

- The terminal object in FinSet is the one element set $\{*\}$. FinSet has all finite limits, and thus has all pullbacks.
- FinSet has all finite colimits, and in particular has all finite coproducts. The quotient of an object X by a subgroup $G \subset \mathrm{Aut}(X)$ is the set of orbits of X under G.
- Any function can be written as the composition of a surjection and an injection, which are precisely the epics and monics in Set. Given an injection $m : X \to Y$, take $Z := Y \setminus m(X)$ and the inclusion $q : Z \hookrightarrow Y$.

Since F is the identity functor, the other three axioms are verified trivially.

Example 3.4. Let π be a profinite group. Then $(\mathrm{FinSet}(\pi), U : \mathrm{FinSet}(\pi) \to \mathrm{FinSet})$, where U is the forgetful functor, form a Galois category.

Before stating the main theorem of this section, we develop more theory revolving around the fundamental functor F. In particular, we will use F to construct a profinite group.

Let $\mathcal{C}$ be a *small* Galois category with fundamental functor F. Let $\mathrm{Aut}(F)$ denote the group of natural isomorphisms $F \to F$. For any $X \in \mathcal{C}$, let $S_{F(X)}$ denote the finite group of permutations of $F(X) \in \mathrm{FinSet}$. Since providing a natural isomorphism $F \to F$ is the data of an isomorphism (bijection) $F(X) \to F(X)$ for each $X \in \mathcal{C}$, we may then consider $\mathrm{Aut}(F)$ as a subgroup

$$\mathrm{Aut}(F) \subset \prod_{X \in \mathcal{C}} S_{F(X)}$$

Here, the smallness assumption on $\mathcal{C}$ plays a key role, if $\mathcal{C}$ were not small, the product above need not exist. Endow each $S_{F(X)}$ with the discrete topology, and $\prod_X S_{F(X)}$ with the product topology. For each morphism $f : Y \to Z$ in $\mathcal{C}$, the set

$$(\sigma_X) \in \prod_X S_{F(X)} : \sigma_Z F(f) = F(f)\sigma_Y$$

is closed, and thus $\mathrm{Aut}(F)$ is a closed subgroup of $\prod_X S_{F(X)}$ by the definition of a natural isomorphism. It follows that $\mathrm{Aut}(F)$ is a profinite group. Finally, we may weaken the smallness assumption by only requiring that $\mathcal{C}$ be *equivalent* to a small category, such a category will be called **essentially small**.
Let X be an object in $\mathcal{C}$. Then $\mathrm{Aut}(F)$ acts continously on the finite set $F(X)$, giving X the structure of an $\mathrm{Aut}(F)$-set.

Let $H(X)$ denote the set $F(X)$ equipped with the $\mathrm{Aut}(F)$-action. Given a morphism $f : X \to Y$, commutativity of the natural isomorphism square

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle \sigma_X} & & \downarrow{\scriptstyle \sigma_Y} \\ F(X) & \xrightarrow[F(f)]{} & F(Y) \end{array}$$

shows that $F(f)$ is a morphism of $\mathrm{Aut}(X)$-sets. Defining $H(f) \coloneqq F(f)$, we see $H : \mathcal{C} \to \mathrm{FinSet}(\mathrm{Aut}(F))$ is a functor. With this in mind, we can now state the main theorem of this section:

Theorem 3.5. *Let $\mathcal{C}$ be an essentially small Galois category, with fundamental functor F. Throuhgout,* $\mathrm{FinSet}(\pi)$ *denotes the category of finite sets equipped with a π-action.*

(1) *The functor $H : \mathcal{C} \to \mathrm{FinSet}(\mathrm{Aut}(F))$ defined above is an equivalence of categories.*

(2) *Let π be a profinite group, and let $U : \mathrm{FinSet}(\pi) \to \mathrm{FinSet}$ be the forgetful functor. Suppose $E : C \to \mathrm{FinSet}(\pi)$ is an equivalence of categories such that the composition $EU = F$, then there is a canonical isomorphism $\pi \cong \mathrm{Aut}(F)$.*

(3) *Any two fundamental functors on $\mathcal{C}$ are isomorphic.*

(4) *If π is a profinite group such that $\mathcal{C}$ and $\mathrm{FinSet}(\pi)$ are equivalent, there is an isomorphism $\pi \cong \mathrm{Aut}(F)$ of profinite groups that is uniquely determined up to inner automorphism of $\mathrm{Aut}(F)$.*

Before beginning the proof of the above theorem, we collect some important definitions and auxillary lemmas that will simplify the proof significantly.

Definition 3.6. Let $\mathcal{C}$ be a Galois category. If X is an object in $\mathcal{C}$, a **subobject** of X is a monomorphism $Y \to X$. Two subobjects $Y \to X$, $Y' \to X$ are isomorphic if there is an isomorphism $Y \cong Y'$ making the following diagram commute.

$$\begin{array}{ccc} Y & \xrightarrow{\sim} & Y' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Properties (4) and (6) imply that $f : Y \to X$ is a monomorphism if and only if $F(f) : F(Y) \to F(X)$ is a monomorphism, thus each subobject $Y \to X$ induces a subset $F(Y) \subset F(X)$. The **intersection** of two subobjects is the pullback $Y \times_X Y' \to X$, where the morphism $Y \times_X Y' \to X$ is monic since both $Y \to X$ and $Y' \to X$ are. Since F preserves pullbacks, $F(Y \times_X Y') = F(Y) \cap F(Y')$, and since F reflects isomorphisms we conclude that two subobjects are isomorphic if and only if $F(Y) = F(Y')$ as subsets of $F(X)$.

Definition 3.7. An object X is connected if it has precisely two subjects, $\mathrm{id} : X \to X$ and $0 \to X$, where 0 is initial. By convention we take that an inital object is not connected.

Next, we establish some important properties of the fundamental functor F. The first is the following proposition.

Proposition 3.8. *Let A be a connected object of a Galois category $\mathcal{C}$, and $a \in F(A)$. For each $X \in \mathcal{C}$, the map*

$$\mathrm{Hom}_{\mathcal{C}}(A, X) \to F(X)$$

sending $f \mapsto F(f)(a)$ is injective.

Proof. Suppose we have two maps $f, g : A \to X$ such that $F(f)(a) = F(g)(a)$. Since F commutes with pullbacks and terminal objects, it commutes with equalizers - thus the equalizer E of f and g is a subobject of A, and $a \in F(E)$. Since A is connected, $C = A$, and thus $f = g$. ☐

Let J be the set of all pairs (A, a), where $A \in \mathcal{C}$ is connected and $a \in F(A)$. Define a partial order : Say $(A, a) \geq (B, b)$ if $b = F(f)(a)$ for some $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ - such a morphism is unique if it exists by the above proposition. If $(A, a) \geq (B, b)$ and $(B, b) \geq (A, a)$, t then (A, a) and (B, b) are the same up to isomorphism. This shows that $\geq$ is indeed a partial ordering on the set of isomorphism classes of J.

We claim that $(J, \geq)$ is filtering. Given $(A, a), (B, b) \in J$, let C be the connected component of $A \times B$ fot which $F(C) \subset F(A) \times F(B)$ contains the pair (a, b). Then $(C, (a, b)) \geq (A, a)$ and $(C, (a, b)) \geq (B, b)$, so J is indeed filtering.

Given $(A, a) \geq (B, b)$ in J, we have the following commutative diagram

$$\begin{array}{ccc} & F(X) & \\ \nearrow & & \nwarrow \\ \mathrm{Hom}_{\mathcal{C}}(B, X) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A, X) \end{array}$$

for all X, which assemble to give a map

$$\varinjlim_{J} \mathrm{Hom}_{\mathcal{C}}(A, X) \to F(X)$$

This map is bijective. Injectivity follows from Proposition 3.8. To see surjectivity, if $x \in F(X)$, then $x \in F(A)$ for some connected component A of X. Thus the map $\mathrm{Hom}(A, X) \to F(X)$ corresponding to the pair (A, x) sends the map $A \to X$ to $x \in F(X)$.

A map $X \to Y$ in $\mathcal{C}$ induces maps $\mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y)$ for all $(A, a) \in J$. These combine to produce the following commutative diagram:

$$\begin{array}{ccc} \varinjlim_J \mathrm{Hom}(A, X) & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ \varinjlim_J \mathrm{Hom}(A, Y) & \longrightarrow & F(Y) \end{array}$$

This shows that F is naturally isomorphic to the functor $\varinjlim_J \mathrm{Hom}(A, -)$. Such a functor F is said to be **prorepresentable**.

Definition 3.9 (Galois Objects)**.** Let A be connected. The following chain of inequalities

$$\# \mathrm{Aut}(A) \leq \# \mathrm{Hom}(A, A) \leq \# F(A)$$

shows that $\mathrm{Aut}(A)$ is finite. Since $\mathcal{C}$ is Galois, the quotient $A/\mathrm{Aut}(A)$ exists. We say A is a **Galois object** if the quotient $A/\mathrm{Aut}(A)$ is the terminal object, which we denote 1. Applying F, we see that A is Galois if and only if the map $F(A)/\mathrm{Aut}(A) \to F(1) = 1$ is an isomorphism. But in FinSet, the quotient is simply the set of orbits, so A is Galois if and only if $\mathrm{Aut}(A)$ acts transitively on $F(A)$. Then clearly we must have $\# \mathrm{Aut}(A) \geq \# F(A)$ - for a connected Galois object we have $\# F(A) = \# \mathrm{Aut}(A) = \# \mathrm{Hom}(A, A)$. Thus $\mathrm{Aut}(A)$ acts freely and transitively on $F(A)$.

Using Galois objects, we have the following strengthening of Proposition 3.8:

Proposition 3.10. *Let X be an object in $\mathcal{C}$. Then there is $(A, a) \in J$, where A is Galois, such that the map $\mathrm{Hom}(A, X) \to F(X)$ (which we already showed is injective) is bijective.*

Proof. We construct the pair (A, a). Let $n := \# F(X)$, and set $Y := X^n = \prod_{i=1}^{n} X$. Let $a \in F(Y) = F(X)^n$ be the element whose j-th coordinate is j for each $j \in F(X)$. Let A be the connected component of Y for which $a \in F(A)$. We check that the pair (A, a) works.
Let $p_j : A \to Y \cong X^n \to X$ be the composition of the natural map $A \to Y$ with the projection onto the j-th

coordinate. The map $\mathrm{Hom}(A, X) \to F(X)$ sends $p_j \mapsto F(p_j)(a) = j$, and is thus surjective. It is already injective by Proposition 3.8, so it is bijective. Thus every morphism $A \to X$ is of the form p_j for some j.
If b is another element of $F(A)$, then the injective map $\mathrm{Hom}(A, X) \to F(X)$ induced by (A, b) is also bijective, since $\#\,\mathrm{Hom}(A, X) = \#F(X)$. Thus the coordinates of b are precisely all the elements of $F(X)$, each occurring once. Thus there is an automorphism σ of Y such that $F(\sigma)$ maps $a \to b$. The automorphism σ maps the connected component A of Y to a connected component B fo Y - since $b \in F(A) \cap F(B)$, we deduce that $A = B$. Thus A has an automorphism sending a to b and is Galois. $\square$

The importance of Galois objects comes from the fact that they form a **cofinal** subset of the partially ordered set J constructed earlier. Here is our definition.

Definition 3.11. Let J be a partially ordered set. A subset $I \subset J$ is said to be **cofinal** if for every $j \in J$, there is some $i \in I$ such that $i \geq j$.

Cofinal subsets are useful since limits can often be computed over them.

Lemma 3.12. *Let J be a partially ordered set that is filtering, and $I \subset J$ a cofinal subset. Then I is filtering, and there is a natural isomorphism*

$$\varinjlim_I A_i \cong \varinjlim_J A_j$$

Proof. This is [19, Theorem IX.3.1] $\square$

Let $I = \{(A, a) \in J : A \text{ is Galois}\}$. We claim that $I \subset J$ is cofinal. Let $(B, b) \in J$. By Proposition 3.10, there exists a connected Galois object A and a map $f : A \to B$. By the connectedness of B, the induced map $F(f) : F(A) \to F(B)$ is surjective, so $F(f)(a) = b$ for some $a \in F(A)$. Thus $(A, a) \in I$ and $(A, a) \geq (B, b)$.
If $f'A \to B$ is another morphism, the surjectivity of $F(f)$ implies there is some $a' \in F(A)$ with $F(f)(a') = F(f')(a)$. Since A is Galois, there is some $\sigma \in \mathrm{Aut}(A)$ such that $a' = F(\sigma)(a)$. Then $F(f\sigma)(a) = F(f')(a)$, so $f\sigma = f$. Thus the action of $\mathrm{Aut}(A)$ on $\mathrm{Hom}(A, B)$ is transitive.

By Lemma 3.12, since $I \subset J$ is cofinal, there is a natural isomorphism $F \cong \varinjlim_I \mathrm{Hom}(A, -)$. Let $(A, a), (B, b) \in I$ be such that $(A, a) \geq (B, b)$, corresponding to a map $f : A \to B$. For any $\sigma \in \mathrm{Aut}(A)$, there is a unique $\tau \in \mathrm{Aut}(B)$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ A & \xrightarrow[f]{} & B \end{array}$$

in particular, it is the automorphism τ such that $F(\tau)(b) = F(f\sigma)(a)$. The map $\mathrm{Aut}(A) \to \mathrm{Aut}(B)$ sending $\sigma \mapsto \tau$ is a surjective group homomorphism by the transitivity proven above. This gives a projective system of finite groups with surjective transition maps. Define $\pi := \varprojlim_I \mathrm{Aut}(A)$, which is a profinite group.

We now define a functor from $\mathcal{C}$ to $\mathrm{FinSet}(\pi)$, the category of finite sets equipped with a continous π-action. Let $X \in \mathcal{C}$ be any object. For each connected Galois object A, the group $\mathrm{Aut}(A)$ acts on $\mathrm{Hom}(A, X)$ via $\sigma \cdot f \mapsto f\sigma^{-1}$. If $(A, a) \geq (B, b) \in I$, this action is compatible with the maps $\mathrm{Aut}(A) \to \mathrm{Aut}(B)$ and $\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$. Thus, we get a continous π-action on $F(X) \cong \varinjlim_I \mathrm{Hom}(A, X)$, which is a finite set.

Write $K(X)$ for the set $F(X)$ with the π-action defined above. Given a morphism $X \to Y$, the induced map $\varinjlim_I \mathrm{Hom}(A, X) \to \varinjlim_I \mathrm{Hom}(A, Y)$ commutes with the π-action. Setting $K(f) := F(f)$ gives a functor $K : \mathcal{C} \to \mathrm{FinSet}(\pi)$.

We first study how the functor K defined above acts on connected objects. So let B be a connected object, and let $(A, a) \in J$ be such that $\mathrm{Hom}(A, B) \cong F(B)$. Let $G \subset \mathrm{Aut}(A)$ be the subgroup

$$G = \{\sigma \in \mathrm{Aut}(A) : f\sigma = f\}$$

for some fixed map $f : A \to B$. Since $\mathrm{Aut}(A)$ acts transitively on $\mathrm{Hom}(A, B)$, there is an isomorphism of π-sets $K(B) \cong \mathrm{Aut}(A)/G$. Since the map $\pi \to \mathrm{Aut}(A)$ is surjective, the action of π on $K(B)$ is transitive. Thus K maps connected objects to connected objects. The fixed morphism $f : A \to B$ above induces a morphism $g : A/G \to B$. It turns out g is an isomorphism. It suffices to show $F(g)$ is an isomorphism. Indeed, since $F(f)$ is surjective, so is $F(g)$. Since the fundamental functor F commutes with quotients, $F(A/G) \cong F(A)/G$, which has cardinality $\# \mathrm{Aut}(A)/G$ since the action of $\mathrm{Aut}(A)$ on $F(A)$ is free and transitive. Since $F(B)$ also has the same cardinality, this shows that $F(g)$, and hence g, are isomorphisms.

Given all this, we can now prove the following:

Lemma 3.13. *The functor K is an equivalence of categories.*

Proof. It suffices to show that K is essentially surjective and fully faithful. We first show essential surjectivity. Every finite π-set is isomorphic to a finite coproduct of transitive π-sets. Since F preserves finite coproducts, so does K. Thus it suffices to consider transitive π-sets, which are all of the form $\mathrm{Aut}(A)/G$ for some connected Galois object A and subgroup $G \subset \mathrm{Aut}(A)$. Then the map $\mathrm{Aut}(A) \to F(A)$ sendng $f \mapsto F(f)(a)$ is a bijection. Thus $K(A)$ is just $F(A)$ equipped with the π-action $\sigma \cdot F(f)(a) \mapsto F(f\sigma^{-1})(a)$. From this we see $K(A) \cong \mathrm{Aut}(A)$, where the π-action on $\mathrm{Aut}(A)$ is defined by $F(f)(a) \mapsto f^{-1}$.
There is also another way $\mathrm{Aut}(A)$ and G act on $K(A)$, namely by $\sigma \cdot x \mapsto F(\sigma)(x)$, which corresponds to right multiplication by σ^{-1}. Thus $K(A)/G \cong \mathrm{Aut}(A)/G$ as π-sets. Since $F(A/G) \cong F(A)/G$, the same holds true for K, and $H(A/G) \cong \mathrm{Aut}(A)/G$ as π-sets. This shows essential surjectivity.
It remains to check full faithfulness, or equivalently, for any $X, Y \in \mathcal{C}$, the map $\psi : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\pi}(K(X), K(Y))$ is bijective. Since F reflects isomorphisms and preserves pullbacks, so does K, and in particular ψ is injective. The next step is to reduce to the case when X and Y are connected. The reduction for X is almost immediate - if $X = \coprod_i X_i$, then $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \coprod_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$, and a similar reduction for $\mathrm{Hom}_{\pi}(K(X), K(Y))$. Let $X \to Y$ be any morphism, and factor it as $X \to Z \to Y$, where $X \to Z$ is epic and $Z \to Y$ is monic. If X is connected, then so is Z, so Z is one of the connected components of Y. Writing $Y = \coprod_j Y_j$, where the Y_j are connected components of Y, we have $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \coprod_j \mathrm{Hom}_{\mathcal{C}}(X, Y_j)$ for connected X. Since $H(X)$ is also connected, we have a similar decomposition for $\mathrm{Hom}_{\pi}(K(X), K(Y))$. So we may assume both X and Y are connected.
Choosing $(A, a) \in I$ sufficiently large, we can write $X = A/G_1$ and $Y = A/G_2$ for some subgroups $G_1, G_2 \subset \mathrm{Aut}(A)$, with $K(X) \cong \mathrm{Aut}(A)/G_1$ and $K(Y) \cong \mathrm{Aut}(A)/G_2$. Any π-homomorphism $\mathrm{Aut}(A)/G_1 \to \mathrm{Aut}(A)/G_2$ is of the form $\tau G_1 \mapsto \tau\sigma G_2$ for some uniquely determined $\sigma G_2 \in \mathrm{Aut}(A)/G_2$. Given σG_2, this homomorphism is well defined if and only if $G_1\sigma \subset \sigma G_2$. This gives the equality

$$\# \mathrm{Hom}_{\pi}(K(X), K(Y)) = \#\{\sigma G_2 \in \mathrm{Aut}(A)/G_2 : G_1\sigma \subset \sigma G_2\}$$

Next, for any $f \in \mathrm{Hom}(X, Y)$, there is some $\sigma \in \mathrm{Aut}(A)$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{h_1} & A/G_1 = X \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{h_2} & A/G_2 = Y \end{array}$$

commute. Choose $a' \in F(A)$ with $F(h_2)(a') = F(fh_1)(a)$, and σ such that $F(\sigma)(a) = a'$. Then $h_2\sigma = h_2\sigma'$ if and only if $\sigma'\sigma^{-1} \in G_2$ if and only if $G_2\sigma = G_2\sigma'$, so f uniquely determines $G_2\sigma$. On the other hand, given $\sigma \in \mathrm{Aut}(A)$, we get a morphism $f : X \to Y$ if and only if $h_2\sigma$ factors via A/G_1, if and only if $h_2\sigma\tau = h_2\tau$ for all $\tau \in G_1$, if and only ig $\sigma G_1 \subset G_2\sigma$. Thus

$$\# \mathrm{Hom}_{\mathcal{C}}(X, Y) = \#\{G_2\sigma : \sigma G_1 \subset G_2\sigma\}$$

Replacing $\sigma \mapsto \sigma^{-1}$, this is the same as $\# \mathrm{Hom}_{\pi}(K(X), K(Y))$. This shows K is fully faithful, and this thus an equivalence of categories.- □

We need one last result before giving the proof of Theorem 3.5.

Lemma 3.14. *Let π be any profinite group, and U : $\mathrm{FinSet}(\pi) \to \mathrm{FinSet}$ the forgetful functor. Then $\mathrm{Aut}(U) \cong \pi$.*

Proof. Since π is profinite, we have an isomorphism $\pi \cong \varprojlim \pi/\pi'$, where π' ranges over all open normal subgroups of π. Each π/π' is automatically a π-set, with the natural action given by left multiplication.
Any natural isomorphism $\sigma : U \to U$ is uniquely determined by bijections $\sigma_X : U(X) \to U(X)$ for $X \in \mathrm{FinSet}(\pi)$. Fix some $X \in \mathrm{FinSet}(\pi)$, and let $x \in U(X)$. Let $x' = \sigma_X(x)$, and π_x be the isotropy group of X in π, which is an open normal subgroup. Without loss of generality, we may assume X is connected, so the π-action on X is transitive. This gives an isomorphism of π-sets $\pi/\pi_x \cong X$, given by sending $\bar{a} \mapsto ax$. This is summarized in the following commutative diagram:

$$\begin{array}{ccc} U(\pi/\pi_x) & \xrightarrow{\sim} & U(X) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \sigma_x} \\ U(\pi/\pi_{x'}) & \xrightarrow[\sim]{} & U(X) \end{array}$$

where τ sending $a\pi_x \mapsto a\pi_{x'}$ is an isomorphism. For all $a \in \pi$ and $x \in X$, we have $a \cdot x = x$ if and only if $a \cdot x' = x'$, where $x' = \sigma_X(x)$. Then $\pi_x = \pi_{x'}$, and so each τ induces a map $\sigma_{\pi/\pi_x} : U(\pi/\pi_x) \to U(\pi/\pi_x)$. Thus σ_X is determined by the $\sigma_{\pi/\pi'}$, where π' ranges over the open normal subgroups of π. We have a natural map $\Phi : \pi/\pi' \to \mathrm{Aut}_\pi(\pi/\pi')$, given by sending $a \mapsto (f_a : b \mapsto ba^{-1})$. We claim this is an isomorphism of groups. To see this is well defined, given $a, a' \in \pi$ such that $a\pi' = a'\pi'$, we have $aa'^{-1} \in \pi'$, and so

$$f_a(b) = ba^{-1} = ba^{-1}aa'^{-1} = ba'^{-1} = f_{a'}(b)$$

By constuction, Φ is a homomorphism and is injective, so it remains to check surjectivity. Choose some $\sigma \in \mathrm{Aut}_\pi(\pi/\pi')$. Fix some $b \in \pi/\pi'$ such that $\sigma(b) = b'$ for some $b' \in \pi/\pi'$. Set $a := b'^{-1}b$. Then $f_a(b) = bb^{-1}b' = b'$. For any $d \in \pi/\pi'$, we have

$$\sigma(d) = \sigma(db^{-1}b) = db^{-1} \cdot b' = da^{-1} = f_a(d)$$

so $\sigma = f_a$ for some a. This shows Φ is surjective, and thus an isomorphism. Any set theoretic map $\pi/\pi' \to \pi/\pi'$ commuting with every $\sigma \in \mathrm{Aut}_\pi(\pi/\pi')$ is given by left multiplication by some element $b \in \pi/\pi'$. So we have

$$\mathrm{Aut}(U) \cong \varprojlim_{\pi'} \mathrm{Aut}(\pi/\pi') \cong \varprojlim_{\pi} \pi/\pi' \cong \pi$$

This completes the proof. □

We finally prove Theorem 3.5.

Proof. We begin by proving the second assertion. Let π be any profinite group, and $H : \mathcal{C} \to \mathrm{FinSet}(\pi)$ any equivalence that when composed with the forgetful functor $U : \mathrm{FinSet}(\pi) \to \mathrm{FinSet}$ gives F, the fundamental functor. Since H is an equivalence of categories, $\mathrm{Aut}(F) \cong \mathrm{Aut}(U)$. By Lemma 3.14, $\mathrm{Aut}(U) \cong \pi$, and so $\pi \cong \mathrm{Aut}(F)$. This shows (2). (1) follows immediately from (2) and the constructions made above. Namely, the group π constructed above and the functor K. Now we show (3). Let $F' : \mathcal{C} \to \mathrm{FinSet}$ be another fundamental functor. By prorepresentability, we can write

$$F \cong \varinjlim_{I} \mathrm{Hom}(A, -)$$

and

$$F' \cong \varinjlim_{I'} \mathrm{Hom}(A, -)$$

where I is the directed set of Galois objects constructed above, and I' is constructed in the exact same manner. Since any two pairs $(A, a_1), (A, a_2) \in I$ sharing the same A are isomorphic, we may retsrict to considering the subsets $I_0 \subset I$, containing exactly one pair (A, a) for each connected Galois object A.
Given (A, a) and (B, b) in I_0 with a morphism $g : A \to B$, there is a unique automorphism β of B such that $F(\beta)(F(g)(a)) = b$. Then $f = \beta g$ satisfies $F(f)(a) = b$, and $(A, a) \geq (B, b)$ in I_0.
From this, we see that $(A, a) \geq (B, b)$ in I_0 if and only if $(A, a') \geq (B, b')$ in I_0'. But the morphisms $f, f' : A \to B$

with $F(f)(a) = b$ and $F(f')(a') = b'$ need not be the same - in fact, for any automorphism $\alpha \in \mathrm{Aut}(A)$, there is an automorphism $\gamma \in \mathrm{Aut}(B)$ that makes the following diagram commute.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} \\ A & \xrightarrow[f']{} & B \end{array}$$

The map sending $\alpha \mapsto \gamma$ gives a projective system of nonempty subsets of $\mathrm{Aut}(A)$, where A ranges over the connected Galois objects. Taking the projective limit, we have simulatenous automorphisms $\alpha_A \in \mathrm{Aut}(A)$ such that all diagrams of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha_A}\downarrow & & \downarrow{\scriptstyle\alpha_B} \\ A & \xrightarrow[f']{} & B \end{array}$$

commute. This gives an isomorphism

$$\varinjlim_{I_0} \mathrm{Hom}(A, -) \cong \varinjlim_{I'_0} \mathrm{Hom}(A, -)$$

which shows $F \cong F'$. This shows (3). Finally, let $H' : \mathcal{C} \to \mathrm{FinSet}(\pi)$ be an equivalence, and let F' denote the composition of H' with the forgetful functor $U : \mathrm{FinSet}(\pi) \to \mathrm{FinSet}$. By (2), we have $\pi \cong \mathrm{Aut}(F')$, then by (3), $F' \cong F$, so we have an isomorphism of profinite groups $\mathrm{Aut}(F) \cong \mathrm{Aut}(F')$, determined upto inner automorphism. This completes the proof of Theorem 3.5. □

4 Projectivity

In this section, we recall some basic properties of projective modules. They will serve as an affine model for the general case in the next section.

Definition 4.1. An A module P is said to be **projective** if any of the following equivalent conditions hold:

(1) For every surjective homomorphismm of A-modules $f : M \to N$ and every homomorphism $g : P \to N$, there is a homomorphism $h : P \to M$ such that $fh = g$

(2) Every short exact sequence of A modules $0 \to M \to N \to P \to 0$ splits.

(3) P is a direct summand of a free module - there is another A module Q such that $P \oplus Q$ is free.

Definition 4.2. Let M be an A module. We say M is **flat** if for any short exact sequence $0 \to N \to P \to Q \to 0$ of A modules, the tensored sequence

$$0 \to N \otimes_A M \to P \otimes_A M \to Q \otimes_A M \to 0$$

is also exact. An A algebra B is flat if it is flat as an A module.

Definition 4.3. Let M be an A module. M is **faithfully flat** if the sequence

$$0 \to N \to P \to Q \to 0$$

is exact if and only if the tensored sequence

$$0 \to N \otimes_A M \to\to P \otimes_A M \to Q \otimes_A M \to 0$$

is exact. An A algebra B is faithfully flat if it is faithfully flat as an A module

Proposition 4.4. *Projective modules are flat.*

Proof. Direct summands of free modules are flat. The result follows from characterization (3) in the above definition. □

The following lemmas characterize projective modules locally. We refer to [18, Chapter 4] for proofs.

Lemma 4.5. *Let A be a local ring, and M a finitely generated A module. Then M is projective if and only if it is free.*

Lemma 4.6. *Let A be a ring, and P an A module. The following conditions are equivalent:*

(1) *P is a finitely generated and projective*

(2) *P is finitely presented, and $P_{\mathfrak{m}}$ is free for any maximal ideal $\mathfrak{m} \subset A$.*

(3) *There is a collection $(f_i)_{i\in I}$ of elements of A such that $\sum_{i\in A} Af_i = A$ such that for all i, the A_{f_i} module P_{f_i} is free of finite rank.*

In the following, we assign to each projective A module a positive integer, called the rank.

Definition 4.7. Let P be a finitely generated projective A module. For each $\mathfrak{p} \in \mathrm{Spec}(A)$, the $A_{\mathfrak{p}}$ module $P_{\mathfrak{p}}$ is free. Define

$$\mathrm{rank}(P) \coloneqq \mathrm{rank}_A(P) : \mathrm{Spec}(A) \to \mathbf{Z}$$

assigning to each $\mathfrak{p}$ the rank of $P_{\mathfrak{p}}$ over $A_{\mathfrak{p}}$. This is locally constant, and hence continous. It follows that if $\mathrm{Spec}(A)$ is connected then $\mathrm{rank}(P)$ is constant. We say that P is **faithfully projective** if $\mathrm{rank}(P) \geq 1$.

Definition 4.8. Let B be an A algebra. We say B is a finite projective A algebra if B is finitely generated and projective as an A module.

We can characterize finite projective A algebras with the following lemma:

Lemma 4.9. *Let B be a finite projective A-algebra. Then*

(1) *The unit map $A \to B$ is injective if and only if $\mathrm{rank}_A(B) \geq 1$.*

(2) *The unit map $A \to B$ is surjective if and only if $\mathrm{rank}_A(B) \leq 1$, if and only if the map $B \otimes_A B \to B$ sending $b \otimes b' \to bb'$ is an isomorphism.*

(3) *The unit map $A \to B$ is an isomorphism if and only if $\mathrm{rank}_A(B) = 1$*

Proof. We show the assertions in order.

(1) If $\mathrm{rank}_A(B)(\mathfrak{p}) = 0$, then $B_{\mathfrak{p}} = 0$, so $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is not injective - since localizations are flat we see $A \to B$ is not injective either. Conversely, if $\mathrm{rank}_A(B)(\mathfrak{p}) \geq 1$, then the kernel of $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ annihilates the free non-zero $A_{\mathfrak{p}}$ module $B_{\mathfrak{p}}$, and hence must be zero. But if $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is injective for all $\mathfrak{p}$, then so is $A \to B$.

(2) First suppose $B \otimes_A B \cong B$. Since the rank of a tensor product is the product of the ranks, we see $\mathrm{rank}_A(B)^2 = \mathrm{rank}_A(B)$, and so $\mathrm{rank}_A(B) \leq 1$. Now, suppose $\mathrm{rank}_A(B) \leq 1$. We without loss of generality assume A is local, so $\mathrm{rank}_A(B)$ is constant. If $\mathrm{rank}_A(B) = 0$ then $B = 0$ and $A \to B$ is trivially surjective. If $\mathrm{rank}_A(B) = 1$, then $\mathrm{End}_A(B)$ is free of rank 1 over A, generated by the identity morphism. The map $\psi : B \to \mathrm{End}(B)$ given by $\psi(b)(x) = bx$ is injective, and the composite $A \to B \to \mathrm{End}_A(B)$ is an isomorphism, so $A \to B$ must be surjective. To conclude, if $A \to B$ is surjective, then $B \cong A/I$ for some ideal $I \subset A$, and we have $B \otimes_A B \cong B/IB \cong B$.

(3) Follows immediately from (1) and (2).

□

We conclude this section by defining two special classes of projective algebras, and stating some important results regarding them that will prove to be useful in the next section.

Definition 4.10. An A-algebra B is **faithfully projective** if it is faithfuly projective as an A module.

Definition 4.11. Let B be a finite projective A algebra. The **trace** of an element $b \in B$, denoted $\mathrm{Tr}(b)$ is defined to be the trace of the A-linear map $x \mapsto bx$. Define an A-linear map $\phi : B \to \mathrm{Hom}_A(B, A)$ by $\phi(x)(y) = \mathrm{Tr}(xy)$ for $x, y \in B$. We say B is a **separable** projective A algebra if the map ϕ is an isomorphism.

We conclude this section with the following three useful propositions.

Proposition 4.12. *Let B be a faithfuly flat A algebra, and P an A module. Then P is finitely generated and projective as an A module if and only if $P \otimes_A B$ is finitely generated and projective as a B module.*

Proof. Suppose $P \otimes_A B$ is finitely generated and projective as a B module. Choose a basis of $P \otimes_A B$ of the form $p \otimes 1$ with $p \in P$. This gives an A linear map $A^n \to P$ which is surjectiv when tensored with B, so by faithful flatness this map itself is already surjective. Let Q be the kernel of this map. Then $0 \to Q \otimes_A B \to B^n \to P \otimes_A B \to 0$ is exact, which shows $Q \otimes_A B$ is finitely generated and projective over B. Applying the same argument to Q, we find that Q is finitely generated, and thus P is finitely presented. Now, let M be any A module. We have a natural map

$$\mathrm{Hom}_A(P, M) \otimes_A B \to \mathrm{Hom}_B(P \otimes_A B, M \otimes_A B)$$

We claim that this map is an isomorphism. First, if $P \cong A^m$ for some finite m, both sides are isomorphic to $(M \otimes_A B)^m$, so the map is an isomorphism. In general, take some free resolution $A^m \to A^n \to P \to 0$ of P. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(P, M) \otimes_A B & \longrightarrow & \mathrm{Hom}_A(A^n, M) \otimes_A B & \longrightarrow & \mathrm{Hom}_A(A^m, M) \otimes_A B \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_A(P \otimes_A B, M \otimes_A B) & \longrightarrow & \mathrm{Hom}_A(A^n \otimes_A B, M \otimes_A B) & \longrightarrow & \mathrm{Hom}_A(A^m \otimes_A B, M \otimes_A B)
\end{array}$$

The top row is exact since $\mathrm{Hom}(-, M)$ is right exact and B is flat, and the bottom row is flat for the same reasons. The right two vertical maps are isomorphisms by the argument above, and the map $0 \to 0$ is clearly an isomorphism. By the five lemma, we conclude that the second remaining vertical arrow is an isomorphism, as desired. Now we show projectivity of P. Let $M \to N$ be a surjective map. By flatness, $M \otimes_A B \to N \otimes_A B$ is surjective, and since $P \otimes_A B$ is projective, the map

$$\mathrm{Hom}_B(P \otimes_A B, M \otimes_A B) \to \mathrm{Hom}_B(P \otimes_A B, N \otimes_A B)$$

is surjective. This in turn implies

$$\mathrm{Hom}_A(P, M) \otimes_A B \to \mathrm{Hom}_A(P, N) \otimes_A B$$

is also surjective, which by faithful flatness shows $\mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N)$ is surjective. Thus P is projective. The converse is true of *any* A algebra B. □

Proposition 4.13. *Let B be an A algebra, and C a faithfully flat A algebra such that $B \otimes_A C$ is a projective separable C algebra. Then B is a projective separable A algebra.*

Proof. By Proposition 4.12 B is a finite projective A algebra. So we only need to show that the map $\phi : B \to \mathrm{Hom}_A(B, A)$ is an isomorphism. By faithful flatness of C, we may check this isomorphism after tensoring up to the map

$$\phi \otimes \mathrm{id}_C : B \otimes_A C \to \mathrm{Hom}_A(B, A) \otimes C$$

Identifying $\mathrm{Hom}_A(B, A) \otimes_A C \cong \mathrm{Hom}_C(B \otimes_A C, C)$, the map $B \otimes_A C \to \mathrm{Hom}_C(B \otimes_A C, C)$ is induced by the trace map, and is an isomorphism since $B \otimes_A C$ is separable over C. This finishes the proof. □

Proposition 4.14. *Let B be a projective separable A algebra and $f : B \to A$ a homomorphism of A algebras. Then there is an A algebra C and an isomorphism $B \cong A \times C$ of A algebras such that f is the composition of the isomorphism $B \to A \times C$ and the natural projection $A \times C \to A$.*

Proof. By A linearity of f, there is a unique e in B such that $f(x) = \mathrm{Tr}(ex)$ for all $x \in B$. We claim e is an idempotent, that is $e^2 = e$. From this the splitting described will follow. Immediately we have $\mathrm{Tr}(e) = 1$. Since f is a homomorphism of rings and Tr is linear, we have $\mathrm{Tr}(exy) = \mathrm{Tr}(f(x)ey)$ for all x and y in B. This shows $ex = f(x)e$ since B is separable, and thus e annihilates $\ker(f)$.

From the exact sequence $0 \to \ker(f) \to B \to A \to 0$ we calculate $\mathrm{Tr}(e) = f(e)$, and so $f(e) = 1$. Thus the A linear map sending $1 \mapsto e$ gives an isomorphism $A \oplus \ker(f) \cong B$ of A modules. Taking $x = e$, we have $ex = f(x)e = e^2 = e$. The map $A \oplus \ker(f) \to B$ commutes with component wise multiplication. Taking $C := \ker(f)$ completes the proof. □

A particularly useful corollary is the following:

Corollary 4.15. *Let A be a ring and B a projective separable A algebra. Consider $B \otimes_A B$ as a B algebra via the inclusion into the second factor. Then there is a B algebra C and a B algebra isomorphism $B \otimes_A B \cong B \times C$ such that after composition with the natural projection $B \times C \to B$, gives the map $B \otimes_A B \to B$ given by $b \otimes b' \to bb'$.*

Proof. $B \otimes_A B$ is a projective separable A algebra, and the map $f : B \otimes_A B \to B$ defined by $f(x \otimes y) = xy$ is a homomorphism of B algebras. Applying the above proposition to f, the result follows. □

5 Étale morphisms

In this section, we prove the main theorem of this paper. We begin by discussing some basic properties of finite ´etale morphisms. First, we briefly recall some properties of morphism of schemes.

Definition 5.1. Let $f : Y \to X$ be a morphism of schemes. We say f is **finite and locally free** if there is a cover of X by affine open subsets $U_i = \mathrm{Spec}(A_i)$ such that $f^{-1}(U_i) = \mathrm{Spec}(B_i)$ is affine for each i, and B_i is finitely generated and free as an A_i module.

Proposition 5.2. *Let $f : Y \to X$ be a morphism of schemes. Then f is finite and locally free if and only if for each open affine subset $U = \mathrm{Spec}(A)$ of X, the open subscheme $f^{-1}(U)$ is affine, and $f^{-1}(U) = \mathrm{Spec}(B)$ for some finite projective A algebra.*

Definition 5.3. Let $f : Y \to X$ be a finite and locally free morphism. Given an affine open $U = \mathrm{Spec}(A)$ subset of X, there is a continous rank map $[B : A] : \mathrm{Spec}(A) \to \mathbf{Z}$. The various rank functions agree on their intersections, so they combine to give a continous map $|X| \to \mathbf{Z}$, called the **degree** of f, denoted $\deg(f)$. In particular, if X is connected, $\deg(f)$ is a unique integer.

Using the degree, we can partially characterize finite locally free morphisms.

Proposition 5.4. *Let $f : Y \to X$ be a finite locally free morphism. Then*

(1) *Y is the empty scheme if and only if* $\deg(f) = 0$

(2) *f is an isomorphism if and only if* $\deg(f) = 1$.

(3) *f is surjective if and only if* $\deg(f) \geq 1$, *if and only if for every affine open subset $U = \mathrm{Spec}(A)$ of X, its preimage $f^{-1}(U) = \mathrm{Spec}(B)$ for a faithfully projective A-algebra B.*

Proof. We may assume $X = \mathrm{Spec}(A)$ is affine. Then by Proposition 5.2, $Y = \mathrm{Spec}(B)$ for a finite projective A-algebra B. Then (1) is immediate, and (2) follows from Lemma 4.9 (3). The third statement reduces, again by Lemma 4.9, to the claim that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective if and only if $A \to B$ is injective. Since B is finite over A, the forward direction follows from [1, Theorem 5.10]. Conversely, if $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\mathfrak{q} \in \mathrm{Spec}(B)$ maps to $\mathfrak{p}$, we have $\mathrm{rank}_A(B)(\mathfrak{p}) \neq 0$ since $B_\mathfrak{q} \neq A_\mathfrak{p} \neq 0$, and so $A \to B$ is injective by Lemma 4.9. This completes the proof. □

We now define finite étale morphisms, which play a central role in even the statement of the main theorem.

Definition 5.5. A morphism of schemes $f : Y \to X$ is said to be finite étale if there is a covering of X by affine open subsets $U_i = \mathrm{Spec}(A_i)$ such that for all i, $f^{-1}(U_i) = \mathrm{Spec}(B_i)$ for some free separable A_i algebra B_i. Given two finite étale mor-phisms $f : Y \to X$ and $g : Z \to X$, a morphism from f to g is a morphism of schemes $h : Y \to Z$ such that $f = gh$. This allows us to speak of the *category* of finite étale coverings of some fixed scheme X, which we denote FEt_X.

Remark 5.6. This is actually not the most general definition one can give - in fact, for a morphism of schemes $f : X \to Y$, there are separate notions of étale morphisms and finite morphisms. We briefly mention these in Section 6.

The following proposition gives a characterization of finite étale morphisms in terms of *projective* separable algebras. Thus, the theory developed in the previous section corresponds to an affine theory of finite étale morphisms.

Proposition 5.7. *Let $f : Y \to X$ be a morphism of schemes. f is finite étale if and only if for every open subset $U = \mathrm{Spec}(A)$ of X, $f^{-1}(U)$ is affine of the form $\mathrm{Spec}(B)$ where B is some projective separable A algebra.*

Proof. The result follows immediately from Proposition 5.2 and the fact that $\phi : B \to \mathrm{Hom}_A(B, A)$ is an isomorphism if and only if the induced map $B_\mathfrak{p} \to \mathrm{Hom}_{A_\mathfrak{p}}(B_\mathfrak{p}, A_\mathfrak{p})$ is an isomorphism for all $\mathfrak{p} \in \mathrm{Spec}(A)$. □

The study of finite étale morphisms is greatly simplified by making suitable base changes $W \to X$.

Proposition 5.8. *Let $f : Y \to X$ be an affine morphism of schemes, and $g : W \to X$ be surjective, finite, and locally free. Then $Y \to X$ is finite étale if and only if $Y \times_X W \to W$ is finite étale.*

Proof. This is [18, Proposition 5.8] □

Our next goal is to give a scheme theoretic analog of trivial finite coverings - that is coverings of the form $X \times E \to X$ for some finite discrete space E.

Definition 5.9. A morphism of schemes $f : Y \to X$ is **totally split** if X can be written as the disjoint union of schemes X_n for positive integers n such that for each n, the scheme $f^{-1}(X_n)$ is isomorphic to the disjoint union of n copies of X_n with the natural morphism $X_n \coprod X_n \coprod \cdots \coprod X_n \to X_n$. A totally split morphism is finite étale.

Proposition 5.10. *Let $f : Y \to X$ be a morphism of schemes. Then f is finite étale if and only if f is affine and $Y \times_X W \to W$ is totally split for some surjective, finite and locally free $W \to X$.*

Proof. The forward direction is immediate from Proposition 5.8. To prove the converse, let $f : Y \to X$ be finite étale, and first assume the degree $\deg(f) = n$ is constant. We induct on n by constructing a surjective finite and locally free morphism $W \to X$ such that $Y \times_X W \to W$ is totally split. When $n = 0$, we can take $W = X$. Suppose now that $n > 0$.
We claim the diagonal morphism is an open and closed immersion. First, if $X = \mathrm{Spec}(A)$ is affine, then so is $Y = \mathrm{Spec}(B)$. By Corollary 4.15, we have the splitting $\mathrm{Spec}(B) \coprod \mathrm{Spec}(C) \cong \mathrm{Spec}(B \otimes_A B)$ which shows our claim in the affine case. In general, cover X by affine opens $\mathrm{Spec}(A_i)$ so that Y is covered by affine opens $\mathrm{Spec}(B_i)$. This gives a covering of $Y \otimes_X Y$ by affines of the form $\mathrm{Spec}(B_i \otimes_{A_i} B_i)$, and the claim follows. This allows us to write $Y \times_X Y' = Y \coprod Y'$, where the second projection $Y \times_X Y \to Y$ is finite étale of degree $n-1$ by pullback stability. By the induction hypothesis we find a map $W \to X$ that is surjective finite and locally free satisfying $Y' \times_Y W \to W$ is totally split. We have

$$Y \times_X W \cong Y \times_X Y \times_Y W \cong (Y \coprod Y') \times_Y W \cong (Y \times_Y W) \coprod (Y' \times_Y W)$$

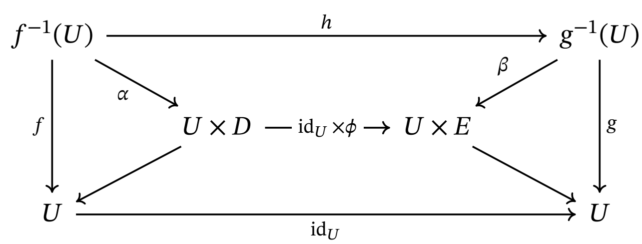
Since $Y' \times_Y W \to W$ and $Y \times_Y W \to W$ are totally split, we also see that $Y \times_X W$ is totally split by the above computation. Since $\deg(f) \geq 1$ we see $Y \to X$ is surjective, and so $W \to X$ is also surjective. Finally, $W \to X$ is finite and locally free since compositions of finite locally free maps are finite locally free. This concludes the induction step.
In the general case where $\deg(f)$ need not be constant, write

$$|X| = \coprod_{n=0}^{\infty} X_n$$

where $|X_n = \{x \in |X| : \deg(f)(x) = n\}$. Setting $Y_n = f^{-1}(X_n) \to X_n$, we see that $Y_n \to X_n$ is finite étale of rank n, so by the above argument there exist surjective finite and locally free morphisms $W_n \to X_n$ such that $Y_n \times_{X_n} W_n \to W_n$ is totally split. Setting $W := \coprod_{n=0}^{\infty} W_n$, the induced map $W \to X$ by the coproduct is our desired map. □

Given a scheme X and a finite set E of cardinality n_E, write $X \times E = \coprod_{i=1}^{n_E} X$. Given a map between two finite sets $E_1 \to E_2$, this induces a natural map $X \times E_1 \to X \times E_2$ which is finite étale.

Lemma 5.11. *Let X, Y, Z be schemes, with totally split morphisms $f : Y \to X$ and $g : Z \to X$. Let $h : Y \to Z$ be such that $f = gh$, and fix some $x \in X$. Then there is an affine open neighbourhood U of X containing x such that f, g and h are trivial above U - there exist finite sets D and E, isomorphisms $\alpha : f^{-1}(U) \to U \times D$ and $\beta : g^{-1}(U) \to U \times E$, and a map $\phi : D \to E$ such that the following diagram commutes:*



Proof. Since this is local, we may assume X is affine, say $X = \mathrm{Spec}(A)$, and that the morphisms f and g are each of constant degree. For a finite set F and ring R, write R^F for the ring of functions $F \to R$, with pointwise addition and multiplication. If f and g are of constant rank, we have $Y = A \times D = \mathrm{Spec}(A^D)$ and $Z = A \times E = \mathrm{Spec}(A^E)$. It then suffices to show that the map of A algebras $\psi : A^E \to A^D$ corresponding to $h : Y \to Z$ is induced by some map $\phi : D \to E$. Consider the local ring A_x, the localization at all powers of x. This ring has no non-trivial idempotents, so the local map $\psi_x : A_x^E \to A_x^D$ is induced by some map $\phi : D \to E$. Thus ψ and $\phi^* A^E \to A^D$ have the same image in $\mathrm{Hom}_A(A_x^D, A_x^E)$, and thus, for $a \in A$ not in the prime ideal corresponding to x, ψ and ϕ^* yield the same map $A_a^E \to A_a^D$. Taking $U = \mathrm{Spec}(A_a)$ finishes the proof. □

Proposition 5.12. *Let $f : Y \to X$ and $g : Z \to X$ be finite étale morphisms of schemes, and $h : Y \to Z$ such that $f = gh$. Then h is finite étale.*

Proof. Follows immediately from the cancellation lemma. □

The following lemma characterizes epimorphisms in the category FEt_X of finite étale coverings of some fixed scheme X.

Lemma 5.13. *Let $f : Y \to X$ and $g : Z \to X$ be two finite étale, and $h : Y \to Z$ a morphism of schemes such that $f = gh$. Then h is an epimorphism in* FEt_X *if and only if h is surjective.*

Proof. First suppose h is an epimorphism. By **??**, the morphism h is finite and locally free, and thus the subscheme

$$Z_0 = \{z \in Z : \mathrm{rank}(h)(z) = 0\}$$

is open and closed in Z. Thus we may decompose $Z = Z_0 \coprod Z_1$, where $Z_1 = Z \setminus Z_0$. We then have that $h^{-1}(Z_0) = \emptyset$ and $h : Y \to Z_1$ is surjective. Thus, the compositions of h with the two natural maps $Z = Z_0 \coprod Z_1 \rightrightarrows Z_0 \coprod Z_0 \coprod Z_1$ are the same, and so the two natural maps must be the same since h is epic. We conclude $Z_0 = \emptyset$, so h is surjective.
Conversely, suppose h is surjective, and let $p, q : Z \to W$ be morphisms such that $ph = qh$, with W finite étale over X. To show $p = q$l, we may work locally, to assume that $X = \mathrm{Spec}(A)$. Then Y, Z, W are also all afine, say $\mathrm{Spec}(B), \mathrm{Spec}(C)$ and $\mathrm{Spec}(D)$ respectively, and the maps p, q, h correspond to maps $D \rightrightarrows C \to B$ composing to give the same map $D \to B$. Since h is surjective, we have that $\mathrm{rank}_C(B) \geq 1$, and so $C \to B$ is injective. Thus the two maps $D \rightrightarrows C$ are the same, and $p = q$. □

We can similarly classify monomorphisms in FEt_X. Here is our result.

Proposition 5.14. *Let $f : Y \to X$ and $g : Z \to X$ be finite étale, and $h : Y \to Z$ be a map such that $f = gh$. Then h is a monomorphism in FEt_X if and only if h is both an open immersion and closed immersion.*

Proof. First, assume $h : Y \to Z$ is a monomorphism. Then the first projection $Y \times_Z Y \to Y$ is an isomorphism, where we note that $Y \times_Z Y$ is finite étale over Z, and hence over X. Let $U = \mathrm{Spec}(A)$ be an affine open. Then $f^1(U)$ and $g^{-1}(U)$ are also affine, say $\mathrm{Spec}(B)$ and $\mathrm{Spec}(C)$ respectively. This gives an isomorphism $B \cong B \otimes_C B$ by sending $b \mapsto b \otimes 1$, so $\mathrm{rank}_C(B) \leq 1$ thus $\mathrm{rank}_Z(Y) \leq 1$. Define

$$Z_n = \{z \in Z : \mathrm{rank}_Z(Y)(z) = n\}$$

, we have $Z = Z_0 \coprod Z_1$. We then have $h^{-1}(Z_0) = \emptyset$ and thus $h : Y \to Z_1$ is an isomorphism. This shows h is both an open and closed immersion. The converse is immediate, which concludes the proof. □

Our next goal is to realize the category FEt_X of finite étale morphisms as a Galois category. In particular, we describe the quotients under finite groups of automorphisms. We begin by considering the category of *affine* morphisms on a fixed scheme X, which we denote Aff_X.

Proposition 5.15. *Let Aff_X be the category of affine morphisms $Y \to X$, with morphisms being defined analogously to FEt_X. Then quotients under finite groups of automorphisms exist in Aff_X.*

Proof. To see this, we will work in the equivalent category of quasi-coherent sheaves of $\mathcal{O}_X$ algebras.

Let $\mathcal{F}$ be a quasicoherent sheaf of $\mathcal{O}_X$ algebras, and G a finite group of automorphisms of $\mathcal{F}$. Given any open subset $U \subset X$, the set $\mathcal{F}(U)^G$ of G-invariants of $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ sub algebra of $\mathcal{F}(U)$ - in particular it is the kernel of the map

$$\mathcal{F}(U) \to \bigoplus_{\sigma \in G} \mathcal{F}(U)$$

sending $a \mapsto (\sigma a - a)_{\sigma \in G}$. Thus, the assignment $U \mapsto \mathcal{F}(U)^G$ is a quasicoherent sheaf of $\mathcal{O}_X$ algebras, which we denote by $\mathcal{F}^G$. Given any morphism of $\mathcal{O}_X$ algebras $f : \mathcal{G} \to \mathcal{F}$ satisfying $\sigma f = f$ for all $\sigma \in G$, we see that f factors uniquely via the inclusion homomorphism $\mathcal{F}^G \to \mathcal{F}$. Translating this back to the category Aff_X, this means that for any affine morphism $f : Y \to X$ and any finite group of automorphisms G, the quotient $g : Y/G \to X$ exists in Aff_X. Explicitly, if $U \subset X$ is any open subset then $g^{-1}(U) \cong f^{-1}(U)/G$ - if $U = \mathrm{Spec}(A)$ is affine with $f^{-1}(U) = \mathrm{Spec}(B)$, then $g^{-1}(U) = \mathrm{Spec}(B)$. □

We want to restrict the above construction to FEt_X. To do so, the following will come in handy.

Proposition 5.16. *Let $Y \to X$ be an affine morphism and G a finite group of automorphisms of $Y \to X$ in Aff_X. Let $W \to X$ be a finite locally free morphism. Then we have an isomorphism $(Y \times_X W)/G \cong (Y/G) \times_X W$ in Aff_W.*

Proof. First, the projection $Y \times_X W \to W$ is affine, and G induces a finite group of automorphisms of $Y \times_X W \to W$ in Aff_W, so the quotient is well defined. The natural morphism $Y \to Y/G$ induces a morphism $g : Y \times_X W \to (Y/G) \times_X W$ that satisfies $g \circ \sigma = g$ for all $\sigma \in G$, and hence induces a map $(Y \times_W)/G \to (Y/G) \times_X W$ by the universal property. To show that this map is an isomorphism, we may work locally, and assume that $X = \mathrm{Spec}(A)$ is affine. Then Y and W are affine as well, say $Y = \mathrm{Spec}(A)$ and $W = \mathrm{Spec}(C)$, with C a finite projective A algebra. We then need to show that the natural map $B^G \otimes_A C \to (B \otimes_A C)^G$ is an isomorphism.
We have an exact sequence $0 \to B^G \to B \to \oplus_{\sigma \in G} B$, where the last map sends $b \mapsto (\sigma(b) - b)_{\sigma \in G}$. Since C is flat, tensoring up we have the exact sequence $0 \to B^G \otimes_A C \to B \otimes_A C \to \oplus_{\sigma \in G} B \otimes_A C$. It follows that $B^G \otimes_A C \cong (B \otimes_A C)^G$, as desired. □

We know restrict the constrution of quotients to the category FEt_X.

Proposition 5.17. *Let $Y \to X$ be a finite étale morphism, and G a finite group of automorphisms of $Y \to X$ in FEt_X. Then the quotient $Y/G \to X$ of $Y \to X$ by G exists in FEt_X.*

Proof. Since étale morphisms are affine, the quotient $g : Y/G \to X$ exists in Aff_X. Thus we only need to verify that $Y/G \to X$ is finite étale.
We first consider the case that $f : Y \to X$ is totally split. Then we may cover X by open sets U such that both $f^{-1}(U) \to U$ and the action of G on U are trivial, that is $f^{-1}(U) \cong U \times E$ for some finite set E, and the action of G on $U \times E$ is induced by some action of G on E. Let E/G be the set of orbits of E under G. Then $U \times (E/G)$

is a quotient of $U \times E$ under G in Aff_U, and thus $U \times (D/G) \cong f^{-1}(U)/G$. Thus $U \times (D/G) \cong g^{-1}(U)$, and $g^{-1}(U) \to U$ is finite étale, which implies $g : Y/G \to X$ is finite étale. For the general case, we choose a surjective, finite, and locally free morphism $W \to X$ for which $Y \times_X W \to W$ is totally split. Then the proposition above shows $(Y \times_X W)/G \to W$ is finite étale and $(Y \times_X W)/G \cong (Y/G) \times_X W$, and thus $Y/G \to X$ is finite étale. □

The following is an étale version of Proposition 5.16.

Proposition 5.18. *Let $Y \to X$ be a finite étale morphism, G a finite group of automorphisms of $Y \to X$ in FEt_X, and $Z \to X$ any morphism of schemes. Then we have an isomorphism $(Y \times_X Z)/G \cong (Y/G) \times_X Z$ in FEt_Z.*

Proof. We have a natural map $(Y \times_X Z)/G \to (Y/G) \times_X Z$ as in the proof of Proposition 5.16. First, assume $Y = X \times D$ for some finite set D, with the action of G on Y induced by some action of G on D. Then $Y \times_X Z \cong Z \times D$ and $(Y \times_X Z)/G$ and $(Y/G) \times_X Z$ are both isomorphic to $Z \times (D/G)$. Next, consider the case where $Y \to X$ is totally split. Then we may cover X by open sets U such that the G-action is trivial on each U. By the above case, the morphism $(Y \times_X Z)/G \to (Y/G) \times_X Z$ is locally an isomorphism, and is thus an isomorphism. Finally, for the general case, choose a surjective, finite, and locally free morphism $W \to X$ such that $Y \times_X W \to W$ is totally split. Then by the above case we have an isomorphism $(Y \times_X W \times_W Z \times_X W)/G \cong (Y \times_X W)/G \times_W Z \times_X W$. Thus, we have an isomorphism

$$(Y_W \times_W Z_W)/G \cong (Y \times_X Z)/G \times_Z W_Z$$

where $(-)_W = (-) \times_X W$ and similarly for $(-)_Z$. Since $Y_W/G \cong (Y/G)_W$, we have

$$(Y_W/G) \times_W Z_W \cong (Y/W)_W \times_W Z_W \cong ((Y \times_X Z)/G) \times_Z W_Z$$

So the map $(Y \times_X Z)/G \to (Y/G) \times_X Z$ becomes an isomorphism after applying $(-) \times_Z W_Z$, and must have already been an isomorphism. □

Recall that the data of a Galois category required a specified functor called the fundamental functor. We will now define such a functor for the category FEt_X.

Definition 5.19. Let X be a scheme. A **geometric point** of X is a morphism $x : \mathrm{Spec}(\Omega) \to X$, where Ω is an algebraically closed field. Geometric points exist if X is non-empty. Let X be a scheme, and $x : \mathrm{Spec}(\Omega) \to X$ a geomtric point. If $Y \to X$ is finite étale, then so is $Y \times_X \mathrm{Spec}(\Omega) \to \mathrm{Spec}(\Omega)$. This induces a functor $H_x : \mathrm{FEt}_X \to \mathrm{FEt}_{\mathrm{Spec}(\Omega)}$ given by $H_x(Y) = Y \times_X \mathrm{Spec}(\Omega)$. Since the absolute Galois group of Ω is trivial (it is its own separable closure), we have an equivalence of categories $J : \mathrm{FEt}_{\mathrm{Spec}(\Omega)} \to \mathrm{FinSet}$. Let $F_x \coloneqq J \circ H_x$.

Theorem 5.20. *Let X be a conected scheme, and x a geometric point of X. The pair (FEt_X, F_x) forms a Galois category.*

Proof. We check the six axioms in order. Since F_x was defined as the composition of an equivalence with H_x, it suffices to check the axioms on H_x.

(1) The identity morphism is a terminal object in FEt_X. If $Y \to W \leftarrow Z$ are morphisms of finite étale coverings, then $Y \times_X W \to Z$ is finite étale by pullback stability, and hence so is $Y \times_W Z \to X$. Thus fiber products exist in FEt_X.

(2) If $Y_i \to X$ is finite étale for $1 \le i \le n$, then so is $\coprod_{i=1}^n Y_i \to X$. Thus FEt_X has finite coproducts, and in particular, $\emptyset \to X$ is an initial object. As we constructed in Proposition 5.17, quotients by finite groups of automorphisms exist.

(3) Let $h : Y \to Z$ be a morphism of finite coverings of X. Write $Z = Z_0 \coprod Z_1$ where $Z_0 = \{z \in Z : \mathrm{rank}_Z(Y) = 0\}$ and $Z_1 = Z \setminus Z_0$ are both open and closed. Since $h^{-1}(Z_0) = \emptyset$, it factors as $Y \to Z_1 \to Z$, where $Y \to Z_1$ is surjective, and thus an epimorphism, and $Z_1 \to Z$ is a monomorphism. Thus any morphism in FEt_X is an epimorphism followed by a monomorphism. Furthermore, any monomorphism is an isomorphism with a direct summand, by Proposition 5.14.

(4) Clearly, H_x takes the terminal object $X \to X$ of FEt_X to the terminal object $\mathrm{Spec}(\Omega) \to \mathrm{Spec}(\Omega)$ of $\mathrm{FEt}_{\mathrm{Spec}(\Omega)}$. Also $H_x = (-) \times_X \mathrm{Spec}(\Omega)$ commutes with pullbacks since this is true for any base change

(5) Any base change commutes with finite coproducts, and transforms epimorphisms into epimorphisms, and commutes with passages to a quotient by a finite group of automorphisms by Proposition 5.18.

(6) By the connectedness assumption on X, the degree of any finite étale covering $Y \to X$ is constant, and is equal the degree of $H_X(Y)$ over $\mathrm{Spec}(\Omega)$. Furthermore, from the equivalence in Theorem 2.16, we conclude $\#F_x(Y) = \deg(Y \to X)$ for any finite étale covering $Y \to X$. Now let $h : Y \to Z$ be a morphism such that $F_x(h) : F_x(Y) \to F_x(Z)$ is a bijection. We will show h is an isomorphism. Factor $Y \to Z$ as in (3) to $Y \to Z_1 \to Z_0 \coprod Z_1$, where $Y \to Z_1$ is surjective. Then the map $F_x(Z_1) \to F_x(Z_0) \coprod F_x(Z_1)$ is also surjective, and thus $F_x(Z_0) = \emptyset$, so $Z_0 = \emptyset$. Thus $Z_1 = Z$ and $Y \to Z$ is surjective. Since $\deg_X(Y) = \#F_x(Y) = \#F_x(Z) = \deg_X(Z)$, so $Y \to Z$ is an isomorphism.

This completes the proof. □

We now prove the main theorem of this section.

Theorem 5.21. *Let X be a connected scheme. Then there is a profinite group π, determined uniquely up to isomorphism, such that the category FEt_X of finite étale coverings of X is equivalent to the category $\mathrm{FinSet}(\pi)$ of finite sets equipped with a continous π action. The profinite group π is called the* ***étale fundamental group*** *of the scheme X, denotes $\pi_1(X)$.*

Proof. For any (connected) scheme X, the category FEt_X is essentially small. The result follows from Theorem 3.5 and Theorem 5.20. Uniqueness in particular comes from Theorem 3.5 (4). □

With F_x, X and x as above, we say $\pi_1(X, x) := \mathrm{Aut}(F_x)$ is the **étale fundamental group** of X in x. In algebraic topology, the fundamental group is functorial, and we may also realize the étale fundamental group as a functor. To see this, let $\mathbf{S}$ be the category where objects are pairs (X, x) of a scheme X and a geometric point x, and morphisms $(X', x') \to (X, x)$ are given by maps of schemes $f : X' \to X$ such that $f \circ x' = x$. Given such a morphism f, the functor $G := (-) \times_X X' : \mathrm{FEt}_X \to \mathrm{FEt}_{X'}$ satisfies $F_{x'} \circ G = F_x$. It follows that we have a continous homomorphism of profinite groups $\pi_1(X', x') \to \pi_1(X, x)$, and therefore $\pi_1(-, -)$ is a functor from $\mathbf{S}$ to the category of profinite groups.

6 Applications

In this section, we compute some examples of étale fundamental groups. First, we give a geometric interpretation of the absolute Galois group of a given field. This will essentially follow immediately from the theory developed in Section 2. Here is our result:

Theorem 6.1. *Let k be a field, and k_s denote a separable closure of k. Then we have an isomorphism*

$$\pi_1(\mathrm{Spec}(k)) \cong \mathrm{Gal}(k_s/k)$$

Proof. From the definition of a finite étale morphism, we see there is an equivalence of categories $\mathrm{SAlg}_k \cong \mathrm{FEt}_{\mathrm{Spec}(k)}$. The former is equivalent to the category of finite sets with a continous $\mathrm{Gal}(k_s/k)$-action by Theorem 2.16, so by Theorem 5.21, we conclude $\pi_1(\mathrm{Spec}(k)) \cong \mathrm{Gal}(k_s/k)$. □

Example 6.2. Take $k = \mathbf{Q}$ above. Then the separable closure of $\mathbf{Q}$ is its algebraic closure, so we conclude $\pi_1(\mathrm{Spec}(\mathbf{Q})) \cong \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

As mentioned in Remark 5.6, we have not yet defined étale morphisms in general. Before giving the full definition, we need some preliminaries.

Definition 6.3. A ring homomorphism $f : A \to B$ is **flat** if it presents B as a flat A module. A morphism $g : Y \to X$ of schemes is **flat** if for every $y \in Y$, the induced ring homomorphism $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat.

Proposition 6.4. *Let $f : A \to B$ a ring homomorphism. The following are equivalent:*

(1) *f is flat.*

(2) *For every prime* $\mathfrak{q}$ *of* B*, the induced map by localization* $A_{f^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ *is flat.*

(3) *The induced morphism* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is flat.*

(4) *For every maximal ideal* $\mathfrak{n}$ *of* B*, the induced map by localization* $A_{f^{-1}(\mathfrak{n})} \to B_{\mathfrak{n}}$ *is flat.*

Proof. See [18, Proposition 6.2]. □

Proposition 6.5. *Let* $f : Y \to X$ *be a morphism of schemes. The following are equivalent:*

(1) f *is flat.*

(2) *For any pair of affine open subsets* $V = \operatorname{Spec} B \subset Y$ *and* $U = \operatorname{Spec} A \subset X$ *with* $f(V) \subset U$*, the induced ring homomorphism* $A \to B$ *is flat.*

(3) *There is a covering of* Y *by open affine subsets* $V_i = \operatorname{Spec} B_i$ *such that for each* i*, there is an open affine subset* $U_i = \operatorname{Spec} A$ *of* X*, with* $f(V_i) \subset U_i$ *for which the induced ring homomorphism* $A_i \to B_i$ *is flat.*

(4) *For every closed point* $y \in Y$*, the induced homomorphism* $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ *is flat.*

Proof. Follows immediately from Proposition 6.4. □

Definition 6.6. Let $f : X \to Y$ be a morphism of schemes that is locally of finite type. The morphism f is said to be unramified at $y \in Y$ if $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ is a finite separable extension of $\mathcal{O}_{X,x}/\mathfrak{m}_x$, where $x = f(y) \in X$. A morphism $f : Y \to X$ is said to be **unramified** if it is locally of finite type and unramified at all $y \in Y$.

If X is a locally noetherian scheme, the notion of a finite étale morphism is equivalent to a morphism being finite and étale. In general however, this need not be the case. For example, let $A = \prod_{i\in I} k_i$ be an infinite product of fields. Define $\mathfrak{p} = \{(x_i)_{i\in I} : x_i = 0 \text{ for all but finitely many } i \in I\}$. Then the morphism $\operatorname{Spec} A/\mathfrak{p} \to \operatorname{Spec} A$ is finite and étale, but not finite étale.

6.1 Normal Schemes

Now we turn to *normal* schemes - schemes where all local rings are integrally closed domains. Our next goal will be to describe finite étale coverings of normal integral schemes. To this end, let X be a normal integral scheme. Given a finite étale covering $Y \to X$, we may uniquely deompose Y as the disjoint union $Y = \coprod Y_i$, where each Y_i is connected. Therefore it suffices to consider connected finite étale coverings.

Let K be the function field of X. Then for any nonempty open $U \subset X$, we may consider $\mathcal{O}_X(U)$ as a subring of K. Let L be a finite separable extension of K. For a nonempty open subset $U \subset X$, let $\mathcal{A}(U)$ be the integral closure of $\mathcal{O}_X(U)$ in L, and let $\mathcal{A}(\emptyset) = \{0\}$. Then $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. This corresponds to an affine morphism $Y \to X$ with $Y = \operatorname{Spec} \mathcal{A}$. We say Y is the **normalization** of X in L. We say X is unramified in L if the morphism $Y \to X$ is unramified.

We state without proof the following result, which allows us to classify finite étale coverings of an integral normal scheme.

Theorem 6.7. *Let* X *be an integral normal scheme with function field* K*, and let* L *be a finite separable extension of* K*. The normalization of* X *in* L *is a connected finite étale covering of* X*. Moreover, every connected finite étale covering of* X *arises in this way.*

Proof. See [18, Theorem 6.13]. □

Theorem 6.7 allows us to express fundamental groups of schemes in terms of Galois groups. Here is our result.

Corollary 6.8. *Let* X *be an integral normal scheme,* K *the function field of* X*, and* $\bar{K}$ *a fixed algebraic closure of* K*. Let* M *be the composite of all finite separable field extensions* L *of* K *with* $L \subset \bar{K}$ *for which* X *is unramified in* L*. Then we have an isomorphism* $\pi(X) \cong \operatorname{Gal}(M/K)$*.*

Proof. The natural morphism $\operatorname{Spec} K \to X$ induces a functor $G : \mathrm{FEt}_X \to \mathrm{FEt}_{\operatorname{Spec} L}$ by $G(Y) = Y \times_X \operatorname{Spec} K$. If L is a finite separable extension of K, the functor G sends the normalization of X in L to $\operatorname{Spec} L$. Thus, the image of G is contained in the full subcategory $\mathcal{D}$ of $\mathrm{FEt}_{\operatorname{Spec} K}$ spanned by objects of the form $\operatorname{Spec} B$, where B is a finite dimensional K algebra that is split by M - that is, we have $B \otimes_K M \cong M \times M \times \cdots \times M$ as M-algebras. This subcategory is equivalent to $\mathrm{FinSet}(\mathrm{Gal}(M/K))$ [18, Ex 2.29]. Thus, G induces a continous group homomorphism $\mathrm{Gal}(M/K) \to \pi_1(X)$.

We claim that the induced homomorphism is bijective. By Theorem 6.7, G sends connected objects to connected objects, and therefore the map $\mathrm{Gal}(M/K) \to \pi_1(X)$ is surjective by [18, Ex 3.23(a)]. We now show injectvity. Let X' be a connected object of the subcategory $\mathcal{D}$ defined above. Then $X' = \operatorname{Spec} L$ for some finite separable field extension of K contained in M. It follows that there are finite field extensions $L_1, \ldots, L_n$ of K contained in M such that X is unramified in each L_i and L is contained in the composite $L_1 \cdot L_2 \cdots L_n$. Let Y_i be the normalization of X in L_i, and $Y \coloneqq Y_1 \times Y_2 \cdots \times Y_n$. Then $Y \in \mathrm{FEt}_X$ and we have $G(Y) = \operatorname{Spec}(L_1 \otimes_K L_2 \otimes_K \cdots \otimes_K L_n)$. The natural surjective map $L_1 \otimes_K \cdots \otimes_k L_n \to L_1 \cdots L_n$ sending $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n$ shows that $\operatorname{Spec}(L_1 \cdot L_2 \cdots L_n)$ is a connected component of $G(Y)$, and the inclusion $L \subset L_1 \cdots L_n$ induces a morphism $\operatorname{Spec}(L_1 \cdots L_n) \to \operatorname{Spec} L$ in $\mathrm{FEt}_{\operatorname{Spec} K}$. It follows from [18, Ex 2.23(b)] that $\mathrm{Gal}(M/K) \to \pi_1(X)$ is injective, and hence bijective. □

6.2 Schemes of Dimension One

We can use the above result to compute many examples. For now, we consider locally noetherian schemes of dimension one.

Example 6.9. Let $X = \operatorname{Spec}(\mathbf{Z}_p)$ for some prime p. Then the function field is $\mathbf{Q}_p$. Let M be the maximal unramified extension of $\mathbf{Q}_p$. Then $\mathrm{Gal}(M/K) \cong \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p) \cong \hat{\mathbf{Z}}$ [24, Section 3-2], so the above theorem implies $\pi_1(\operatorname{Spec} \mathbf{Z}_p) \cong \hat{\mathbf{Z}}$.

Example 6.10. Let K be any number field, and A the ring of integers of K. Consider $X \coloneqq \operatorname{Spec} A[1/a]$, where $a \in A$ is some nonzero element. The closed points $x \in X$ are in bijective correspondence with the nonzero prime ideals of A that do not divide a, and M is the maximal algebraic extension of K that is unramified at these primes. Thus, we find $\mathrm{Gal}(M/K) \cong \pi_1(\operatorname{Spec} A[1/a])$.

Example 6.11. Continuing from Example 6.10, consider the case $A = \mathbf{Z}$ and $a = 1$, so $X = \operatorname{Spec}(\mathbf{Z})$. By Minkowski's theorem, if the discriminant of a number field $L \neq \mathbf{Q}$ is greater than 1, then L ramifies at *some* prime. Thus, the maximal unramified extension is $\mathbf{Q}$, and $\pi_1(\operatorname{Spec} \mathbf{Z}) \cong \mathrm{Gal}(\mathbf{Q}/\mathbf{Q}) \cong 1$ is trivial.

In this final section, we compute the fundamental groups of $\mathbf{P}^1$ and $\mathbf{A}^1$ over a field K. First, we recall some basic facts about valuations.

Definition 6.12. Let K be an field. An **exponential valuation** is a map $v : K^\times \to \mathbf{Z}$ satisfying

- $v(xy) = v(x) + v(y)$
- $v(x + y) \geq \min\{v(x), v(y)\}$

The **trivial valuation** maps $x \mapsto 0$ for all $x \in K^\times$.

Let f be a monic polynomial in $K[t]$ for some field K. We define an exponential valuation v_f on $K(t)$ by defining $v_f : K(t)^\times \to \mathbf{Z}$ as follows: Any element $a/b \in K(t)$ can be written uniquely as $f^n g/h$, where $f \nmid h$. We take $v_f(a/b)$ to be this n. This clearly satisfies the properties of being a valuation, and it restricts to a trivial valuation on K. Similarly, define $v_\infty : K(t)^\times \to \mathbf{Z}$ by $v_\infty(g/h) = \deg g - \deg h$. Again, this is an exponential valuation which restricts to a trivial valuation on K. In fact, every nontrivial valuation on $K(t)$ that is trivial on K is equivalent to one of the valuations mentioned above.

Definition 6.13. Let v be one of the above valuations, and let F be a finite separable extension of $K(t)$. We may extend v to a valuation w on F. Completing $K(t)$ at v, and F at w, we have a finite extension $F_w/K(t)_v$. We say that w is **tamely ramified** if the extension is separable, and the ramification $e(w/v)$ is not divisible by the characteristic of K. If $e(w/v) = 1$, then w is **unramified** over v. We say v is tamely ramified (resp. unramified) if every w extending v is tamely ramified (resp. unramified).

Proposition 6.14. *Let K be a field, and F a finite separable extension of $K(t)$ such that every element of $F \setminus K$ is transcendental over K. Furthermore, suppose that the valuation v_∞ is tamely ramified in F, and all the valuations v_f are unramified. Then $F = K(t)$*

Proof. See [18, Proposition 6.20] □

This has the following useful corollary.

Corollary 6.15. *Let K be a field, and F a finite separable extension of $K(t)$. Suppose that v_∞ is tamely ramified in F and all v_f's are unramified in F. Then $F = L(t)$ for some finite separable extension L of K.*

Proof. Let $L = \{x \in F : x \text{ is algebraic over } K\}$. Then t is transcendental over L. Applying Proposition 6.14 to $L(t) \subset F$, we see $L(t) = F$. Since $K(t) \subset L(t)$ is finite and separable, the same is true for $K \subset L$. □

We can now compute the fundamental groups of projective and affine lines. We begin with $\mathbf{P}^1_K$.

Example 6.16. Let K be a field, and $X := \mathbf{P}^1_K$. The function field of X is simply $K(t)$, and the valuations of $K(t)$ corresponding to the closed points of X are the valuations v_f and v_∞. If F is a finite separable extension that is unramified at these valuations, then Corollary 6.15 implies $F \subset K_s(t)$, where K_s is a separable closure of K. We also have that $K(t) \subset K_s(t)$ is unramified at all of these valuations, so the maximal unramified extension is $K_s(t)$. By Corollary 6.8, we have

$$\pi_1(\mathbf{P}^1_K) \cong \mathrm{Gal}(K_s(t)/K(t)) \cong \mathrm{Gal}(K_s/K) \cong \pi_1(\mathrm{Spec}\, K)$$

In particular, we see that $\pi_1(\mathbf{P}^1_K)$ is trivial if and only if K is separably closed.

Example 6.17. Finally, we compute $\pi_1(\mathbf{A}^1_K)$ for a field K of characterstic zero. Again, the function field is $K(t)$, and the valuations corresponding to the closed points of X are the valuations v_f. v_∞ in this case. Since the characteristic of K is zero, all valuations on $K(t)$ are trivial on K. In particular, we see that v_∞ is tamely ramifies in any finite extension $K(t) \subset F$. Therefore, similar to the case of the projective line, the maximal unramified etension M of $K(t)$ is $K_s(t)$, and we see $\pi_1(\mathbf{A}^1_K) \cong \mathrm{Gal}(K_s/K)$.

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